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WAVES DUE TO A SUBMERGED BODY

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LIST OF SYMBOLS

A_i	Airy function
f	Depth of a point source
$F(\theta)$	Expression defined in Equation (7.1)
g	Acceleration of gravity
H	Depth of keel of submarine
k_o	$= U^2/g$
K_o, K_1	Modified Bessel Functions of 2nd kind
$L_n^{(1)}, L_n^{(2)}$	Integrals defined in Equation (19) and (20), respectively
M	Strength of point source
M_1	Strength of line source per unit length
N	$= k_o R$
p_m, q_m	Coefficients of assymptotic expansion defined in Equation (A9)
Q	$= \sqrt{(1 - 8 \tan^2 \delta)}$
R	Radial distance from singularity defined in Equation (2)
U	Relative velocity of water at infinity
x, y, z	Rectangular right hand coordinates, x in the direction of U and z vertically upward
x_1	x coordinate of singularity
δ	Angle defined in Equation (2)
δ_c	Critical angle of δ defined in Equation (8.1)
θ_1, θ_2	Constant defined in Equation (8)

μ	Fictitious frictional force
$\mu(\delta), \nu(\delta)$	Coefficients of transformation defined in Equation (A4)
ϕ	Velocity potential
ω	Expression defined in Equation (2)
ζ	Wave height due to a point source
ζ_r, ζ_ℓ	Regular wave and local disturbance due to a point source respectively
ζ_1	Wave height due to a source line
$\zeta_{1r}, \zeta_{1\ell}$	Regular wave and local disturbance due to a source line respectively
ζ_{rt}, ζ_{rd}	Transverse and divergent waves, respectively.

INTRODUCTION

The pattern and the height of surface waves due to a surface ship or a submerged body has been considered by many hydrodynamists since Lord Kelvin (1891) worked out the wave system due to a pressure point on the surface. This problem is attractive not only because of practical applications in connection with the wave resistance or stability of ships but also as a basic physical phenomenon and because of the delicate application of mathematics involved.

Kelvin calculated the integral representing the wave height by the method of stationary phase which is reasonably valid far behind the pressure point inside the critical angle ($\pm 19^{\circ}\ 28'$). Havelock (1908) evaluated the wave height on the critical line. Hogner (1923) investigated the wave height due to a Kelvin source in the vicinity of the critical line. Peters (1949) and Ursell (1960) improved the theory and the result using the method of steepest descent.

Havelock investigated the wave due to a submerged sphere (1928) [see also Wigley (1930)], and the infinite draft ship (1932). Inui and Kinnaka also investigated waves of the infinite draft ship and performed the numerical evaluation (1957).

In this paper, the wave height due to a submerged point source and a source line is investigated. The wave height consists of two parts; the regular wave and the local disturbance. The method of evaluating the integral representing regular waves is different in two regions, that in the vicinity and that far behind the singularity. In the vicinity of the singularity, the numberical integration is performed by the method of Gauss' quadrature. At the far place, the method of stationary phase is used. The behavior of the wave height near the critical line is discussed.

For the local disturbance, the integral can be evaluated in the closed form immediately above the singularity. A scheme involving numerical computation by high speed machine is worked out for other points.

As an example, the wave height due to a simple form of submarine with given dimensions is calculated in detail. The influence of the submarine sail is included.

WAVE HEIGHT DUE TO A SIMPLE SOURCE

As usual, the water is assumed to be inviscid, homogeneous and incompressible. Hence there exists a potential Φ. The coordinate system is fixed in the singularity and only the steady problem is considered. The origin of the right handed coordinate system O-xyz is located on the mean free surface; x is directed along the uniform relative velocity of the water, and z is vertically upward. The surface wave is assumed to be small compared with the wave length, and the principle of superposition holds.

The derivation of the formula of the wave height ζ due to a point source located at $(x_1, 0, -f)$, with the strength M is well known (e.g. see Wigley 1949).

$$\zeta = \frac{U}{g}\Phi_x = -\frac{M}{2\pi U}\,\mathrm{Re}\int_{-\pi}^{\pi}\int_{0}^{\infty}\frac{i2\,k_o k \sec\theta\, e^{k(i\omega - f)}}{k-k_o\sec^2\theta - i\mu\sec\theta}\,dk\,d\theta \qquad (1)$$

where

$$\omega = (x - x_1) \cos \theta + y \sin \theta$$
$$= R \cos (\theta - \delta), \text{ with } x - x_1 = R \cos \delta, \; y = R \sin \delta \qquad (2)$$

$$k_o = \frac{g}{U^2} \;, \; \delta = \text{arc tan} \left(\frac{y}{x - x_1} \right) \qquad (3)$$

μ is the fictitious frictional force which is to be put zero after the evaluation of the integral. By co[illegible] integration (see Appendix 1) ζ becomes the sum of two integrals,

$$\zeta = \zeta_r + \zeta_\ell \qquad (4)$$

$$\zeta_r = 4 \frac{M}{U} k_o \int_{-\pi/2 + \delta}^{\pi/2} \exp (-k_o f \sec^2 \theta) \sec^3 \theta \cos (k_o \omega \sec^2 \theta) d\theta \qquad (5)$$

$$\zeta_\ell = - \frac{2M}{\pi U} \int_{-\pi/2 + \delta}^{\pi/2 + \delta} \int_0^\infty \frac{\exp (-m\omega)}{k_o^2 \sec^4 \theta} \frac{m \sec \theta}{m^2} \times$$

$$\times \{k_o \sec^2 \theta \sin (mf) - m \cos (mf)\} \, dm d\theta$$

$$= - \frac{2M}{U\pi f} \int_{-\pi/2}^{\pi/2} \int_0^\infty \frac{\exp (-m \frac{R}{f} \cos \theta) \, m \sec \theta_o}{k_o^2 f^2 \sec^4 \theta_o + m^2} \{k_o f \sec^2 \theta_o \sin m$$

$$- m \cos m\} \, dm d\theta \qquad (6)$$

where $\theta_o = \theta + \delta$, ζ_r is called the regular wave and ζ_ℓ is the local disturbance which diminishes with large x.

This is the fundamental formula for the surface wave since the body in the water is usually represented by source distributions. The wave height due to a given body is then obtained by superposition of all the waves produced by each source. The wave due to higher order singularities such as a doublet is the derivative of Equation (4) with respect to the position in the direction of the higher order singularity.

The evaluations of ζ_r and ζ_ℓ in Equations (5) and (6) are not simple. These will be discussed separately in the following two sections.

EVALUATION OF THE REGULAR WAVE

The evaluation of the integral ζ_r of Equation (5) is different in the two regions, near and far behind the singularity.

(A) Stationary Phase. On the surface far behind the singularity, i.e. when k_oR is large, the method of stationary phase can be applied to the integral (5). This method was originated by Lord Kelvin (1887) and can be found discussed in any related text (e.g. Lamb 1945, p. 395).

Let us consider the integral

$$\zeta_r \approx 4\,M\,k_o \int_{-\pi/2}^{\pi/2} \exp\left(-k_o f \sec^2 \theta\right) \sec^3 \theta \cos(N F(\theta))\, d\theta \quad (7)$$

where $F(\theta) = \sec^2 \theta \cos(\theta - \delta)$, $N = k_o R$. (7.1)

Since N is large, the part of integration between $-\pi/2$ and $-\pi/2 + \delta$ is negligible.

The root of the derivative of $F(\theta)$ with respect to θ is obtained as

$$F'(\theta) = 0$$

or

$$2 \tan^2 \theta + \tan\theta \tan\delta + 1 = 0$$

$$\left.\begin{matrix} \tan\theta_1 \\ \tan\theta_2 \end{matrix}\right\} = -\frac{1}{4}\left\{\cot\delta \mp \sqrt{(\cot^2\delta - 8)}\right\} \qquad (8)$$

which is only significant when $\cot^2\delta - 8 \geq 0$ or $\delta \leq \tan^{-1}\frac{1}{2\sqrt{2}} = \delta_c$

(8.1)

Now (7) can be evaluated as the sum of two waves, ζ_{rt} transverse, ζ_{rd} divergent,

$$\zeta_r = \zeta_{rt} + \zeta_{rd}$$

$$\left.\begin{aligned} \zeta_{rt} &= 4k_o \frac{M}{U} \sec^3\theta_1 \exp(-k_o f \sec^2\theta_1) \sqrt{\frac{2\pi}{N|F''(\theta_1)|}} \cos\left(N F(\theta_1) \pm \frac{\pi}{4}\right) \\ \zeta_{rd} &= 4k_o \frac{M}{U} \sec^3\theta_2 \exp(-k_o f \sec^2\theta_2) \sqrt{\frac{2\pi}{N|F''(\theta_2)|}} \cos\left(N F(\theta_2) \pm \frac{\pi}{4}\right) \end{aligned}\right\} (9)$$

where $$F''(\theta) = (6 \tan^2 \theta + 1) \sec^2 \theta \cos(\delta-\theta) + 4 \sec^2 \theta \tan \theta \sin(\delta-\theta) \tag{10}$$

and the sign of $\pm \pi/4$ is decided as the sign of $F''(\theta)$.

However, this is only for $\delta < \delta_c$. When $\delta = 0$, which means the centerline, $\theta_1 = 0$, $\theta_2 = -\pi/2$,

and $$F''(\theta_1) = 1$$
$$F''(\theta_2) = -\infty$$

Hence

$$\zeta_r = \zeta_{rt} = 4k_o \frac{M}{U} \exp(-k_o f) \sqrt{\frac{2\pi}{N}} \cos\left(N + \frac{\pi}{4}\right) \tag{11}$$

Since $N \equiv k_o R \equiv \frac{gR}{U^2}$, the wave length on the centerline is

$$2\pi U^2/g \tag{11.1}$$

When $\delta = \delta_c$ which is the critical line, $\theta_1 = \theta_2$, $F''(\theta_1) = 0$ and this makes Equation (9) singular. When $\delta > \delta_c$ the right hand side of Equation (8) becomes imaginary. In the vicinity of this critical line far behind the singularity, Hogner (1923), Peters (1949), and Ursell (1960) investigated thoroughly the situation for the wave due to a pressure point.

Putting $\tan \theta = u$ in Equation (5) and neglecting the interval $(-\pi/2, -\pi/2 + \delta)$, we obtain

$$\zeta_1 = 8k_o \frac{M}{U} \exp(-k_o f) \int_0^\infty \sqrt{(1 + u^2)} \exp(-k_o f u^2) \times$$

$$\times \cos[k_o t \sqrt{(1 + u^2)}] \cos(k_o y u \sqrt{(1 + u^2)}) du$$

$$= 4k_o \frac{M}{U} \exp(-k_o f) \mathrm{Re} \int_{-\infty}^\infty \sqrt{(1 + u^2)} \exp(-k_o f u^2) \times$$

$$\times \exp[iN\{(\cos\delta - u \sin\delta)\sqrt{(1 + u^2)}\}] du \quad (12)$$

where $k_o x = N \cos \delta \qquad k_o y = N \sin \delta$.

We note that the expression for the wave height due to the pressure point on the surface is exactly the same as that due to the submerged doublet with the corresponding strength and zero depth (compare Equation(2.2) of Ursell (1960) with Havelock's (1928) Equation (12) at far behind the singularity by the change of variable as above) . We mentioned already the relation between the wave heights due to a doublet and a point source.

The method of Chester, Friedman, and Ursell (1957) can be adopted for the asymptotic expansion of the above equation in the vicinity of the critical line as in Ursell's paper (1960). The idea is that in order to use the method of steepest descent in the vicinity of the critical line(where the usual method of steepest descent fails) the integral is represented in a series of Airy functions (see Jeffreys and Jeffreys, 1946 § 17.07) by the regular transformation of the variable of integration. Chester et al (1957) worked out the rigorous theory. The only difference between the two cases of the pressure point (by Ursell (1960)) and the submerged point source is in the form of the analytic function g(z) in the integral

$$\int g(u) \exp [Nf(u)] \tag{12a}$$

(of Chester et al. Equation (1.1)).

The way of deriving the asymptotic expansion of Equation (12) in the vicinity of the critical angle is exactly the same as Ursell's (1960). His coefficient of transformation $\mu(\delta)$ and $\nu(\delta)$ are exactly the same for each angle, which are tabulated in his paper. The coefficients of the asymptotic expansion of p_o and q_o are different. These are derived in Appendix 2 keeping the close connection with Ursell's paper (1960). The result is:

$$p_o(\delta) = \frac{3^{\frac{3}{4}}}{2^{\frac{3}{2}}} \frac{\cos \delta}{\sin^{\frac{3}{2}} \delta} \left(\frac{\mu(\delta)}{1-8 \tan^2\delta}\right)^{\frac{1}{4}} [\exp\{- \frac{k_of}{16} \cot^2\delta (1+Q)^2\} (1+Q)^{\frac{3}{4}} (1- \frac{1}{3}Q)^{\frac{3}{4}}$$

$$+ \exp\{- \frac{k_of}{16} \cot^2\delta \; (1-Q)^2\} (1-Q)^{\frac{3}{4}} (1 + \frac{1}{3}Q)^{\frac{3}{4}}] \tag{13}$$

$$q_o(\delta) = \frac{3^{\frac{3}{4}}}{2^{\frac{7}{3}}} \frac{\cos\delta}{\sin^{\frac{3}{2}}\delta} \left(\frac{\mu(\delta)}{1-8\tan^2\delta} \right)^{\frac{1}{4}} Q^{-1} [\exp[-\frac{k_o f}{16}\cot^2\delta(1-Q)^2](1+Q)^{\frac{3}{4}}(1-\frac{1}{3}Q)^{\frac{3}{4}}$$

$$- \exp[-\frac{k_o f}{16}\cot^2\delta\ (1-Q)^2](1-Q)^{\frac{3}{4}}(1+\frac{1}{3}Q)^{\frac{3}{4}}] \qquad (14)$$

where $Q = \sqrt{(1-8\tan^2\delta)}$.

Thus we obtain the asymptotic expansion of the wave height taking the real part of the integral (12) (instead of imaginary part as in Ursell's (1960) Equation (3.6))

$$\zeta_1 \sim 4k_o \frac{M}{U} \exp(-k_o f)[\frac{p_o(\delta)}{N^{\frac{1}{3}}} A_i(-N^{\frac{2}{3}}\mu(\delta)) \cos(N\nu(\delta)$$

$$- \frac{q_o(\delta)}{N^{\frac{2}{3}}} A_i^1 (-N^{\frac{2}{3}}\mu(\delta)) \sin(N\nu(\delta)] \qquad (15)$$

which is valid in some finite angle including $\delta = \delta_c$ by the theory proved by Chester et al. (1957). The terms neglected in the assymtotic series (15) are of order $N^{-\frac{4}{3}} A_i$, and $N^{-\frac{5}{3}} A_i'$ at most. (See Appendix 2)

As was pointed out previously, the difference between the wave height due to a pressure point and a doublet is only in the depth effect. Hence, for the case of a submerged doublet, p_o and q_o (in Ursell's table) should only be multiplied by corresponding exponential factor, $\exp[-k_o f\ (1 + u_+^2)]$ and

$\exp\{-k_o f (1 + u_-^2)\}$ respectively. Since $u_+ = u_- - \frac{\cot \delta}{4}$ on the line $\delta = \delta_c$, the same factor $\exp\{-k_o f (1 + \frac{\cot \delta}{4})\}$ applied to p_o and q_o would be sufficient in the very close vicinity of the critical line. Namely the wave pattern will be the same in the close vicinity of the critical line except that the wave height is reduced relative to that of the pressure point. However, the wave height due to a submerged point source, Equation (15) is different not only in p_o and q_o but also the sine and cosine are exchanged from that of the pressure point (Ursell's (4.12)).

(B) In the Vicinity of the Singularity-

Equation (5) can be written as

$$\zeta_1 = 4k_o \frac{M}{U} \int_{-\pi/2+\delta}^{\pi/2} \sec^3\theta \exp(-k_o f \sec^2\theta) \cos(k_o \omega \sec^2\theta) d\theta$$

$$= 4k_o \frac{M}{U} \int_{-\pi/2}^{\pi/2} \sec^3\theta \exp(-k_o f \sec^2\theta) \cos(k_o \omega \sec^2\theta) d\theta$$

$$- 4k_o \frac{M}{U} \int_{\pi/2-\delta}^{\pi/2} \sec^3\theta \exp(-k_o f \sec^2\theta) \cos[k_o R \cos(\theta + \delta) \sec^2\theta] d\theta \tag{16}$$

For the first integral on the right hand side of the above equation, a useful table is available from the Admiralty Research Laboratory (1956). This is derived from the numerical evaluation of the related integral by the method of Wilson (1957). The second term is negligible when $x - x_1$ is large, but in the near field this term may be important. If a high speed calculating machine is used, ζ_1 for reasonably small R/f can be readily evaluated by the use of Gauss' mechanical quadrature formula.

If we change the variable θ by $\tan\theta = u$ as in Equation (12), Equation (16) becomes

$$\zeta_1 = 4k_oM \exp(-k_of)\int_{-\infty}^{\infty}\sqrt{(1 + u^2)}\exp(-k_ofu^2)\cos[k_ox\sqrt{(1 + u^2)}]\, X$$

$$X \cos[k_oyu\sqrt{(1 + u^2)}]\, du$$

$$- 4k_oM \exp(-k_of)\int_{x/y}^{\infty}\sqrt{(1 + u^2)}\exp(-k_ofu^2)\, X$$

$$X \cos[k_oR\cos(\text{arc tan } u \pm \delta)(1 + u^2)]\, du \qquad (17)$$

with $x_1 = 0$ for convenience.

The first integral of the right hand side can be evaluated by the Hermite-Gauss quadrature formula for which the weights (Christofel numbers) and the zeros of the Hermite polynomials calculated by

Kopal are available up to the order 20. For the second integral, if x is small, the limit of the integral may better be changed to

$$\int_{\frac{x}{y}}^{\infty} = \int_{0}^{\infty} - \int_{0}^{\frac{x}{y}}$$

When x/y is sufficiently large by the change of variable

$$t = u^2$$

the second integral of the right hand side of Equation (17) will become

$$2\,\frac{M}{Uf}\exp\left[-k_o f\left(1 + \frac{x^2}{y^2}\right)\right]\int_0^{\infty}\sqrt{(1 + t)}\exp(-t_1)\ \times$$

$$\times \cos\left[k_o \bar{R}\cos(\tan^{-1}\sqrt{t} + \delta)(1 + t)\right]\frac{dt}{\sqrt{t}} \tag{18}$$

where $t = \dfrac{t_1}{k_o f} + \dfrac{x^2}{y^2}$.

This can be evaluated by Laguerre-Gauss quadrature for which the weights and the zeros of the Laguerre polynomials obtained by Zalzer and Zucker (1949) are available up to the order 15.

Of course, by using a high speed machine such as the IBM 7090, the other method of direct integration of (17) even with

Simpson's rule is applicable to the extent that the oscillation is rapid and the amplitude is small enough to be neglected. But the speed of calculation also highly depends on the magnitude of R/f. The advantage of the method of Gauss' quadrature lies in its simplicity and in time saving, but the error is extremely difficult to estimate.

EVALUATION OF THE LOCAL DISTURBANCE

The evaluation of the integral (6) representing the local disturbance is also extremely complicated. There has been no analytical form of the result. Even the numerical method is not easy, especially in the vicinity directly above the singularity, because of the slow convergence of the integral. Numerical evaluation of equation (6) was obtained for a certain range of parameters by Wigley(1949). The evaluation can be treated separately in two regions - in the vicinity of the singularity and away from it.

(A) In the Vicinity of the Singularity

At the point immediately above the singularity the integral can be evaluated analytically in closed form. The expression (6), involves a type of integral

$$L_n^{(1)}(p,q,s) = \int_0^{\infty} \frac{m^{2n}\, e^{-mp} \cos mq}{s^2 + m^2}\, dm \qquad (19)$$

and

$$L_n^{(2)}(p,q,s) = \int_0^{\infty} \frac{m^{2n+1}\, e^{-mp} \sin mq}{s^2 + m^2}\, dm \qquad (20)$$

for $$p > 0,\ n > 0$$
$$p \geq 0,\ n = 0$$

which reduce to the well known Laplace integrals (See Erdelyi 1953) when $p = n = 0$

$$L_0^{(2)}(o,q,s) = \frac{\pi}{2} e^{-qs} \tag{21}$$

Fortunately a part of the integral (6), is exactly in this form when $R = 0$ or $\omega = 0$. The other part is of the form $L_1^{(1)}(o,q,s)$ which is the divergent integral. However this can be evaluated as the limit of $R \rightarrow 0$, which can not only be derived formally but also can be justified rigorously without difficulty.

$$L_1^{(1)}(o,q,s) = \lim_{p \rightarrow 0} \int_0^{\infty} \frac{m^2\ e^{-mp} \cos mq}{s^2 + m^2}\, dm$$

$$= \lim_{p \rightarrow 0} \frac{d}{dq} \int_0^{\infty} \frac{m\ e^{-mp} \sin mq}{s^2 + m^2}\, dm$$

$$= -\frac{\pi s}{2}\ e^{-qs} \tag{22}$$

From Equations (19) - (22),

$$\int_0^{\infty} \frac{mk_0 \sec^3\theta \sin (mf)}{k_0^{\,2} \sec^4 \theta + m^2}\, dm = \frac{\pi}{2} k_0 \sec^3 \theta \exp (-k_0 f \sec^2 \theta)$$

$$\lim_{R \to 0} \int_0^{\infty} \frac{\exp(-mR \cos\theta) m^2 \sec\theta \cos(mf)}{k_o^2 \sec^4 \theta + m^2}\, dm = -\frac{\pi}{2} k_o \sec^3\theta \exp(-k_o f \sec^2\theta)$$

Hence from (6)

$$\zeta_\ell = -\frac{2}{R}\frac{M}{U} \lim_{R \to 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\infty} \frac{\exp(-mR \cos\theta) m \sec\theta_o}{k_o^2 \sec^4 \theta_o + m^2} \, X$$

$$X \, [k_o \sec^2 \theta_o \sin(mf) - m \cos(mf)] \, dm d\theta$$

$$= -2k_o \frac{M}{U} \int_{-\pi/2}^{\pi/2} \sec^3(\theta + \delta) \exp[-k_o f \sec^2(\theta + \delta)] \, d\theta$$

where $\theta_o = \theta + \delta$, but if $\delta = 0$,

$$\zeta_\ell = -k_o \frac{M}{U} \exp(-k_o f/2) \, [K_o(k_o f/2) + K_1(k_o f/2)] \qquad (23)$$

where K_o, K_1 are modified Bessel functions of the 2nd kind. (See, e.g. Lunde 1952, page 33).

It is easy to see that ζ_ℓ is antisymmetric with respect to the y axis. Hence, $\zeta_\ell = 0$ on $x = 0$ or $\delta = \pi/2$. Besides, ζ_ℓ has a discontinuity in appearance on the line $y = 0$. Physically this is impossible. However, it is obvious in equation (6) tha

ζ_ℓ is a continuous function of δ even when $R \to 0$, taking the value between ζ_ℓ $(R \to 0,\ \delta = 0)$ and ζ_ℓ $(R \to 0,\ \delta = \pi)$ -
$-\zeta_\ell$ $(R \to 0,\ \delta = 0)$. This means that, even on $R \to 0$, the water surface is continuous although the derivative is not so except in the direction $\delta = \pi/2$. Of course, we have to remember that we neglected the viscosity and the surface tension.

When R is very small but not equal to zero, there is no easy method to integrate, even numerically. However, it may be worth while mentioning Barakat's work (1961) on integrating L functions and the use of it here.

Using the notations (19) and (20), Equation (6) can be written,

$$\zeta_\ell = -\frac{2M_1}{nfU}\int_{-\pi/2}^{\pi/2}[k_o f \sec^3\theta_o \, L_o^{(2)}\{\frac{R}{f}\cos\theta, 1, k_o f \sec^2\theta_o\}$$

$$-\sec\theta_o L_o^{(1)}\{\frac{R}{f}\cos\theta, 1, k_o f \sec^2\theta_o\}]\, d\theta \qquad (24)$$

where $\theta_o = \theta + \delta$.

Barakat changed the L function which has an oscillating integrand into other non-oscillating integrals plus known functions, i.e.

$$L_o^{(1)}(p,q,s) = \frac{(ps)}{2s}\, e^{-qs}\int_0^{qs}\frac{e^{t}}{(ps)^2+t^2}\,dt - \frac{(ps)}{2s}\, e^{qs}\int_0^{qs}\frac{e^{-t}}{(ps)^2+t^2}\,dt$$

$$+\frac{1}{s}\,[C_1(ps)\sin(ps) + \{\frac{\pi}{2} - S_1(ps)\}\cos(ps)]\, e^{-qs}$$

(25)

$$L_o^{(2)}(p,q,s) = \frac{(ps)}{2} e^{-qs} \int_o^{qs} \frac{e^t\,dt}{(ps)^2 + t^2} + \frac{(ps)}{2} \int_o^{qs} \frac{e^{-t}\,dt}{(ps)^2 + t^2}$$

$$+ [C_1(ps) \sin(ps) + (\frac{\pi}{2} - S_1(ps)) \cos(ps)]e^{-qs} \qquad (26)$$

where C_1 and S_1 are cosine and sine integrals respectively, and

$$L_1^{(1)}(p,q,s) = - s^2 L_o^{(1)} + \frac{p}{p^2 + q^2} \qquad (27)$$

(See Barakat (1961) Equations (27), (28) and (29)).

If we use Equations (24) - (27), the integrand of Equation (24) can be evaluated at certain points. Then the usual method of integration can be used with respect to θ.

B. When R/f is not too small - the integral (6) converges reasonably well. We may then use a combination of quadrature methods. Equation (6) can be written,

$$\zeta_{\ell} = - \frac{2M}{2\pi f U} \int_{-\pi/2}^{\pi/2} \sum_{n=0}^{\infty} \int_{0}^{2\pi} \frac{\exp(-m_o \frac{R}{f} \cos \theta)\, m \sin \theta_o \{k_o f \sec^2\theta_o \sin m - m_o \cos m\}}{k_o^2 f^2 \sec^4 \theta_o + m_o^2}\, dm d\theta$$

where $m_o = m + 2n\pi$, $\theta_o = \theta + \delta$

$$= - \frac{2M}{\pi f U} \int_{-\pi/2}^{\pi/2} \sum_{n=0}^{\infty} \int_{-\pi/2}^{\pi/2} \Bigg[\frac{\exp[(-m_1 + 2n\pi) \frac{R}{f} \cos \theta](m_1 + 2n\pi)}{k_o^2 f^2 \sec^4 \theta_o + (2n\pi + m_1)^2}$$

$$- \frac{\exp[-(m_1 + (2n + 1)\pi] \frac{R}{f} \cos \theta][m_1 + (2n + 1)\pi}{k_o^2 f^2 \sec^4 \theta_o + [(2n + 1)\pi + m_1)^2} \; k_o f \sec^3\theta_o \cos m \, dm d\theta$$

$$- \frac{2M}{\pi f U} \int_{-\pi/2}^{\pi/2} \sum_{n=0}^{\infty} \int_{-\pi/2}^{\pi/2} \Bigg[\frac{\exp[-(m_1 + 2n\pi) \frac{R}{f} \cos \theta](m_1 + 2n\pi)^2}{k_o^2 f^2 \sec^4 \theta_o + (2n\pi + m_1)^2} \; -$$

$$- \frac{\exp\left[-\{m_1 + (2n+1)\pi\}\frac{R}{f}\cos\theta\right]\{m_1 + (2n+1)\pi\}^2}{k_o^2 f^2 \sec^4\theta_o + \left[(2n+1)\pi + m_1\right]^2} \Bigg] \sec\theta_o \sin m \, dm d\theta \tag{28}$$

where $m_1 = m + \frac{\pi}{2}$.

Because of the exponential term, the integrand diminishes rapidly when n increases. We may either use Gauss' or Simpson's quadrature formulas between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for both integrals. Especially when $\frac{R}{f}\cos\theta$ is reasonably large, say >.5, we may directly apply the Laguerre-Gauss quadrature formula to the integral with respect to m of Equation (6), and then the Legendre-Gauss or Simpson's quadrature formula to the integral with respect to θ.

In Figure 1, the value of (6) is plotted for each Froude number with respect to the depth versus the distance, x/y, where the curves for Froude numbers 1 and .707 are plotted using the table by Wigley for the sake of comparison.

WAVE DUE TO A SUBMERGED SOURCE LINE

The wave height ζ_1 due to a submerged finite source line can be obtained by integrating equations), (5) and (6) with respect to the depth f from the top f_1 to the bottom f_2. The method of evaluating the integrals involved are exactly similar to the case of a point source. If we non-dimensionalize the physical quantities by the depth f_1 of the top of infinite submerged source line, the result can be used for the case of a finite source line by matching the corresponding parameters.

We write

$$\zeta_1 = \zeta_{1r} + \zeta_{1\ell}$$

where ζ_{1r} is the regular wave and $\zeta_{1\ell}$, the local disturbance.

$$\zeta_{1r} = 8\frac{M_1}{U}\left[\exp(-k_o f)\int_0^\infty \exp(-k_o f\ u^2)(1+u^2)^{-\frac{1}{2}} \cos[k_o x \sqrt{(1+u^2)}]\cos[k_o y\ u \sqrt{(1+u^2)}\right.$$

$$\left.-\tfrac{1}{2}\exp(-k f)\int_{x/y}^\infty \exp(-k_o f\ u^2)(1+u^2)^{-\frac{1}{2}}\cos(k_o R \cos(\arctan u+\delta)(1+u^2))\delta u\right]_{f_2}^{f_1}$$

where M_1 is the strength of the source line per unit length.

$$\zeta_{1\ell} = - \frac{2M_1}{\pi U} \int_{-\pi/2}^{\pi/2} \int_0^{-\infty} \frac{\exp(-mR\cos\theta)\sec\theta_1 (k_o \sec^2\theta_1 \cos mf + m \sin mf)}{k_o^2 \sec^4\theta_1 + m^2} dm d\theta \Bigg]_{f=f_2}^{f=f_1}$$

$$= \left[\frac{2M_1}{\pi U} \int_{-\pi/2}^{\pi/2} \sum_{n=0}^{\infty} \int_{-\pi/2}^{\pi/2} \left\{ \frac{\exp[-(m_1 + 2n\pi)\frac{R}{f}\cos\theta]}{k_o^2 f^2 \sec^4\theta_1} \right.\right.$$

$$\left. - \frac{\exp[-\{m_1 + (2n+1)\pi\}\frac{R}{f}\cos\theta]}{k_o^2 f^2 \sec^4\theta_1 + \{m_1 + \pi(2n+1)\}^2} \right\} k_o f \sec^3\theta_1 \sin m dm d\theta$$

$$- \frac{2M_1}{\pi U} \int_{-\pi/2}^{\pi/2} \sum_{n=0}^{\infty} \int_{-\pi/2}^{\pi/2} \left\{ \frac{\exp[-(m_1 + 2n\pi)\frac{R}{f}\cos\theta]\,(m_1 + 2n\pi)}{k_o^2 f^2 \sec^4\theta_o + (m_1 + 2n\pi)^2} \right.$$

$$\left. - \frac{\exp[-\{m_1 + \pi(2n+1)\}\frac{R}{f}\cos\theta][m_1 + (2n+1)\pi]}{k_o^2 f^2 \sec^4\theta_o + [m_1 + \pi(2n+1)]^2} \right\} \times$$

$$\times \sec\theta_o \cos m dm d\theta \Bigg]_{f=f_2}^{f=f_1} .$$

where $\theta_o = \theta + \delta$, $m_1 = m + \frac{\pi}{2}$

At $R \rightarrow 0$

$$\zeta_{1\ell} = -\frac{2M_1}{U}\int_{-\pi/2}^{\pi/2} \sec\theta_o(\exp(-k_o f_1 \sec^2\theta_o)-\exp(-k_o f_2 \sec^2\theta_o))d\theta$$

where $\theta_o = \theta + \delta$. When $\delta = 0$ and $R \rightarrow 0$.

$$\zeta_{1\ell} = \frac{2M_1}{U}[\exp(-k_o f_2/2)K_o(k_o f_2/2) - \exp(-k_o f_1/2)K_o(k_o f_1/2)]$$

Figures 2 and 3 show the local disturbance for the submerged infinite source line for the different distances x/f_1 versus Froude number with respect to f_1, and for the different Froude numbers versus x/f_1 respectively.

WAVES DUE TO A SUBMARINE

If we represent the submarine hull by a Rankine ovoid and the sail of the submarine by a combination of a source line and a sink line, the wave height due to the submarine can be obtained by the methods described above. In Figure 4, the dimensions of the submarine to be used as an example are shown. Figure 5 shows the corresponding positions and strengths of a point source and a point sink for the hull, and a source line and a sink line for the sail. These are obtained using the theory of Rankine's solids (see e.g. Milne-Thomson page 441) for the hull, and the two dimensional theory (see Milne-Thomson page 203) for the sail, which is approximately valid. All figures related to the submarine are shown with dimensional quantities. Figure 6a shows the amplitude of the regular wave immediately behind and 5 wave lengths behind the stern for several depths. Since the wave length is $2\pi V^2/g$ as shown in Equation (11.1), the interactions between the sources and the sinks of the hull and sail cause the wiggling of the curve. The effect of speed on the wave height is so great that 5 cycle semi-log paper is used to plot the amplitudes.

Figure 6b shows the comparison of the amplitudes due to hull and sail at a distance 5 wave lengths behind the stern. We can see that the sail waves are dominant up to a speed of about 11 knots for each depth, while the hull waves become gradually dominant with increasing speed above 11 knots.

In Figures 7-11 the wave heights near the submarine are plotted for several angles from the bow of the sail or the bow of the hull. The two Figures 7 and 8 of these five figures are transformed to contour diagrams in Figures 12 and 13. As was shown in Equation (15), for large N, the wave in the vicinity of the critical line decays like $N^{-\frac{1}{3}}$ while the wave in $\delta << \delta_c$ decays like $N^{-\frac{1}{2}}$ where $N = k_0 R$. In the vicinity of the submarine, it is noticeable that the transverse wave is first prominent and gradually the divergent wave becomes more prominent when the distance increases.

In Figures 14 - 17 the local disturbance on the centerline due to the sail is plotted. The local disturbance due to a point source or a line source is antisymmetric with respect to the axis perpendicular to the velocity through the projection of the singularity on the mean free surface, as shown before. The local disturbance due to a sink is exactly opposite in sign. Therefore, the local disturbance due to the sail or the hull is exactly symmetric with respect to the midsail or the midship plane. Figure 18 shows the local disturbances in different directions.

In Figures 19 - 22 the local disturbance due to the hull is plotted for different depths and different speeds.

APPENDIX 1

To evaluate

$$- \phi_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\infty} \frac{12k_o k \sec \theta \exp[k (i\omega - f)]}{k - k_o \sec^2 \theta - i\mu \sec \theta} dk d\theta$$

$$\omega = (x - x_1) \cos \theta + y \sin \theta$$

$$= R \cos (\theta - \delta)$$

$$\delta = \text{arc tan} \left(\frac{y}{x - t_1}\right)$$

Note the only singularity at $k = k_o + i\mu \sec \theta$,

where $\sec \theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sec \theta < 0$ for $\frac{\pi}{2} < \theta \leq \pi$, $-\pi < \theta < -\frac{\pi}{2}$

when $\omega > 0$ or $-\frac{\pi}{2} + \delta < \theta < \frac{\pi}{2} + \delta$, we take the contour ABO instead of OA in Figure A1. When $\omega < 0$, or $\frac{\pi}{2} + \delta < \theta < \pi$ and $-\pi < \theta < -\frac{\pi}{2} + \delta$, we take the contour ACO instead of OA. Then the singularity in the case $-\frac{\pi}{2} < \theta < -\frac{\pi}{2} + \delta$ and $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \delta$ is outside the contour. Thus we obtain:

$$-\Phi_x = -\left[\int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}} - \int_{-\pi}^{-\frac{\pi}{2}} - \int_{\frac{\pi}{2}+\delta}^{\pi}\right] 2\,k_o^2 \sec^3 \exp\left[k_o \sec^2\theta\,(i\omega - f)\right] d\theta$$

$$-\frac{1}{2\pi}\int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}+\delta}\int_0^{\infty} \frac{i\,2\,k_o m \sec\theta \exp\left[im(i\omega - f)\right]}{im - k_o \sec^2\theta}\, dm\,d\theta$$

$$= -4\,k_o^2 \int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}} \sec^3\theta \exp\left(-k_o f \sec^2\theta\right) \cos\left(k_o\omega \sec^2\theta\right) d\theta$$

$$-\frac{4}{2\pi}\int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}+\delta}\int_0^{\infty} \frac{m \sec\theta \exp(-m\omega)\left(k_o \sec^2\theta \sin mf - m\cos mf\right)}{k_o^2 \sec^4\theta + m^2}\, dm\,d\theta$$

APPENDIX 2

To apply the method of steepest descent to Equation (12), first Debye's curves through two saddle points are given by Peters (1949) in his Figures 5, 6 and 7. These are the roots of

$$\frac{df}{du}(u,\delta) \equiv \frac{d}{du}\left[(\cos\delta - u\sin\delta)\sqrt{(1+u^2)}\right] = 0 \qquad \text{(A1)}$$

Namely, the points

$$\delta_+(\delta) = \frac{1}{4}\left[\cot\delta + \sqrt{(\cot^2\delta - 8)}\right]$$

and

$$\delta_-(\delta) = \frac{1}{4}\left[\cot\delta - \sqrt{(\cot^2\delta - 8)}\right] \qquad \text{(A2)}$$

By the method of Chester et al. (1957), the integral can be expressed in terms of the Airy function (see Jeffreys and Jeffreys 1946 § 17.07).

$$A_i(Z) = \frac{1}{2\pi i}\int_{\infty\exp(-\frac{1}{3}\pi i)}^{\infty\exp(\frac{1}{3}\pi i)} \exp\left(\frac{1}{3}t^3 - Z\,t\right) dt$$

$$= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + Zt\right) dt \qquad \text{(A3)}$$

where Z is real, we transform the variable u by

$$F(u,\delta) \equiv (\cos\delta - u\sin\delta)\sqrt{(1+u^2)} = -\frac{1}{3}v^3 + \mu(\delta)v + \lambda(\delta) \qquad \text{(A4)}$$

which is proved to be a regular one-to-one transformation if the zeros of Equation (1) , $u_{\pm}$ and the zeros of

$$\frac{2F\ (u,\delta)}{du}\ \frac{du}{dv} = -v^2 + \mu(\delta) = 0 \tag{A5}$$

or

$$v = \pm\ \mu^{\frac{1}{2}} \tag{A6}$$

correspond. Ursell (1960) calculated $\mu(\delta)$ and $v(\delta)$ for each δ near δ_c, which can be used here without any change. Now the analytic function $g(v)\ \frac{du}{dv}$ in Equation (12a) or the corresponding expression in Equation (12)

$$\exp(-k_o f u^2)(1 + u^2)^{\frac{1}{2}}\ du/dv \tag{A7}$$

is expanded in the form

$$\sum_{0}^{\infty} p_m(\delta)(v^2 - \mu(\delta))^m + v\sum_{0}^{\infty} q_m(\delta)(v^2 - \mu(\delta))^m \tag{A8}$$

which holds uniformly when v and $\delta - \delta_c$ are sufficiently small. Coefficients p_m's and q_m's can be found by repeatedly differentiating both sides and putting $u = u_{\pm}$ and $v = \pm\ \mu^{\frac{1}{2}}$. Then the asymptotic expansion of the integral ( ) is shown to be

$$\begin{aligned} &\Sigma\ p_m(\delta) \int (v^2 - \mu(\delta))^m \exp[iN(-\tfrac{1}{3}\ v^3 + \mu v - \nu)]\ dv \\ &+ \Sigma\ q_m(\delta) \int v(v^2 - \mu(\delta))^m \exp[iN(-\tfrac{1}{3}\ v^3 + \mu v - \nu)] dv \end{aligned} \tag{A9}$$

where the integration can be extended with a negligible error from $-\infty \exp(\frac{\pi}{6}i)$ to $\infty \exp(-\frac{\pi}{6}i)$ (cf Peters Figure 6). To obtain the dominant term it is sufficient to calculate the leading coefficients $p_o(\delta)$ and $q_o(\delta)$. Putting $u = u_\pm(\delta)$, $v = \pm \mu^{\frac{1}{2}}(\delta)$ in Equation (19), we obtain

$$\exp(-k_o f\, u^2{}_\pm)(1 + u^2{}_\pm)^{\frac{1}{2}}\left(\frac{du}{dv}\right)_\pm = p_o(\delta) \pm \mu^{\frac{1}{2}}(\delta)\, q_o(\delta) \qquad \text{(A10)}$$

where $\left(\frac{du}{dv}\right)_\pm$ can be obtained as follows:

$$\frac{\delta^2 F}{\delta u^2}\left(\frac{du}{dv}\right)^2 + \frac{\delta F}{\delta u}\frac{d^2u}{dv^2} = -2v$$

whence $$\left(\frac{\delta^2 F}{\delta u^2}\right)_\pm \left(\frac{du}{dv}\right)_\pm^2 = \mp 2\mu^{\frac{1}{2}}$$

If we put $Q \equiv \sqrt{(1 - 8\tan^2 Q)}$, from (A2)

$$u_\pm = \frac{\cot\delta}{4}(1 \pm Q)$$

and $$1 + u^2{}_\pm = \frac{3}{16}\cot^2\delta\,(1 \pm Q)\,(1 \mp \tfrac{1}{3}Q) \qquad \text{(A11)}$$

From (A4) $$\left(\frac{\delta^2 F}{\delta u^2}\right)_\pm = \mp\frac{4}{3^{\frac{1}{2}}}\sin\theta\, Q(1 \pm Q)^{-\frac{1}{2}}(1 \mp \tfrac{1}{3}Q)^{-\frac{1}{2}}$$

Hence $$\left(\frac{du}{dv}\right)_\pm = \frac{3^{\frac{1}{4}}}{2^{\frac{1}{2}}}\,\frac{\mu^{\frac{1}{4}}}{\sin^{\frac{1}{2}}\theta}\,Q^{-\frac{1}{2}}(1 \pm Q)^{\frac{1}{4}}(1 \mp \tfrac{1}{3}Q)^{\frac{1}{4}} \qquad \text{(A12)}$$

From (A11) and (A12)

$$\exp(-k_o f\, u_\pm^2)^{\frac{1}{2}} \left(\frac{du}{dv}\right)_\pm$$

$$= \exp\left[-\frac{k_o f}{16}\cot^2\delta\, (1\pm Q)^2\right] \frac{3^{\frac{3}{4}}}{2^{\frac{5}{2}}} \frac{\cos\delta}{\sin^3\delta} \left(\frac{\mu(\theta)}{1-8\tan^2\delta}\right)^{\frac{1}{4}} (1\pm Q)^{\frac{3}{4}} (1\mp\frac{1}{3}\, Q)^{\frac{3}{4}} \tag{A13}$$

Solving the simultaneous equations in (A10)

$$p_o(\delta) = \frac{3^{\frac{3}{4}}}{2^{\frac{7}{2}}} \frac{\cos\delta}{\sin^3\delta} \left(\frac{\mu(\delta)}{1-8\tan^2\delta}\right)^{\frac{1}{4}} \Big[\exp\left[-\frac{k_o f}{16}\cot^2\delta\, (1+Q)^2\right](1+Q)^{\frac{3}{4}}(1 - \frac{1}{3}Q)^{\frac{3}{4}}$$

$$+ \exp\left[-\frac{k_o f}{16}\cot^2\delta\, (1-Q)^2\right](1-Q)^{\frac{3}{4}}(1 + \frac{1}{3}Q)^{\frac{3}{4}}\Big] \tag{A14}$$

$$q_o(\delta) = \frac{3^{\frac{3}{4}}}{2^{\frac{7}{2}}} \frac{\cos\delta}{\sin^3\delta} \left(\frac{\mu(\delta)}{1-8\tan^2\delta}\right)^{\frac{1}{4}} Q^{-1} \Big[\exp\left[-\frac{k_o f}{16}\cot^2\delta(1+Q)^2\right](1+Q)^{\frac{3}{4}}(1 - \frac{1}{3}Q)^{\frac{3}{4}}$$

$$- \exp\left[-\frac{k_o f}{16}\cot^2\delta\, (1-Q)^2\right](1-Q)^{\frac{3}{4}}(1 + \frac{1}{3}Q)^{\frac{3}{4}}\Big] \tag{A15}$$

where $Q = \sqrt{(1-8\tan^2\delta)}$.

Putting these p_o and q_o into Equation (A9) we obtain the first two terms of the asymptotic expansions of the integral (12)

$$2\pi i \exp(-iN\nu(\delta)) \, [iN^{-\frac{1}{3}} p_o(\delta) A_i(-N^{\frac{2}{3}}\mu(\delta)) + N^{-\frac{2}{3}} q_o(\delta) A_i'(-N^{\frac{2}{3}}\mu(\delta))]$$

Then the assymptotic expansion of ζ_1 is obtained taking the real part

$$\zeta_1 = 4k_o M \exp(-k_o f)\left[\frac{p_o(\delta)}{N^{\frac{1}{3}}} A_i(-N^{\frac{2}{3}}\mu(\delta)) \cos(N\nu(\delta)) - \frac{q_o(\delta)}{N^{\frac{2}{3}}} A_i'(-N^{\frac{2}{3}}\mu(\delta)) \sin(N\nu(\delta))\right] \qquad \text{(A16)}$$

which is valid in some finite angle including $\delta = \delta_c$.

Since (A9) can be rearranged in the form

$$\frac{A_i'(N^{\frac{2}{3}}\mu)}{N^{\frac{1}{3}}} \Sigma \frac{a_n(\mu)}{N^{2n}} + \frac{A_i'(N^{\frac{2}{3}}\mu)}{N^{\frac{5}{3}}} \Sigma \frac{b_n(\mu)}{N^{2n}}$$

$$+ \frac{A_i'(N^{\frac{2}{3}}\mu)}{N^{\frac{2}{3}}} \Sigma \frac{c_n(\mu)}{N^{2n}} + \frac{A_i'(N^{\frac{2}{3}}\mu)}{N^{\frac{4}{3}}} \Sigma \frac{b_n(\mu)}{N^{2n}}$$

(See Chester et al. (1957).

where the first two terms come from the first summation related to p_m of expression (A9) and the other two from the [illegible] related to q_m in (A9). Hence the terms neglected in (A16) by taking only $m = 0$ are of order $N^{-\frac{4}{3}} A_1$ and $N^{-\frac{5}{3}} A_1$ at most.

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$$\int_0^{\pi/2} \exp(-\alpha^2 \sec^2\theta) \sec^n \theta \begin{matrix}\cos \\ \sin\end{matrix} (\beta \sec \theta) d\theta$$

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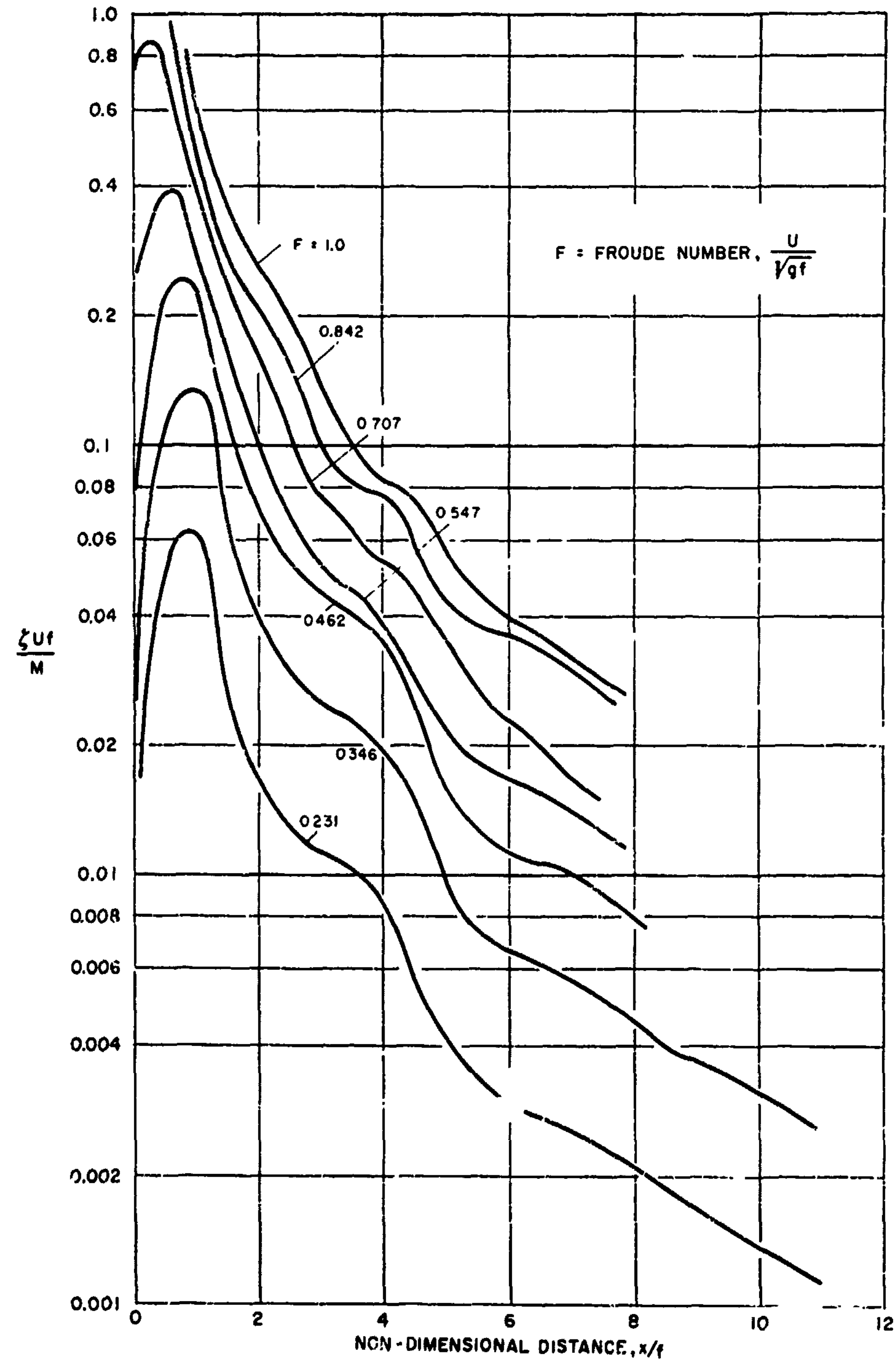

FIGURE 1 – NON-DIMENSIONAL LOCAL DISTURBANCE DUE TO A SUBMERGED POINT SOURCE

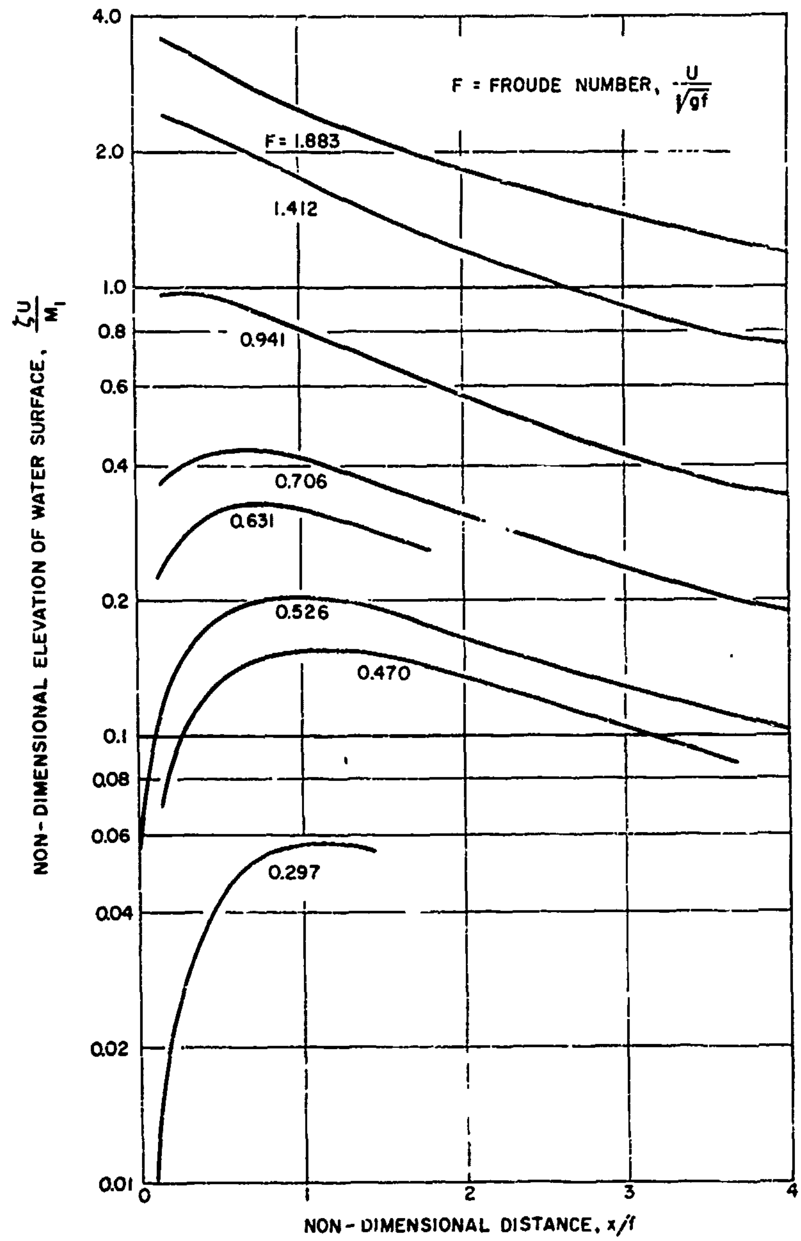

FIGURE 2 - LOCAL DISTURBANCE DUE TO AN INFINITE SOURCE LINE

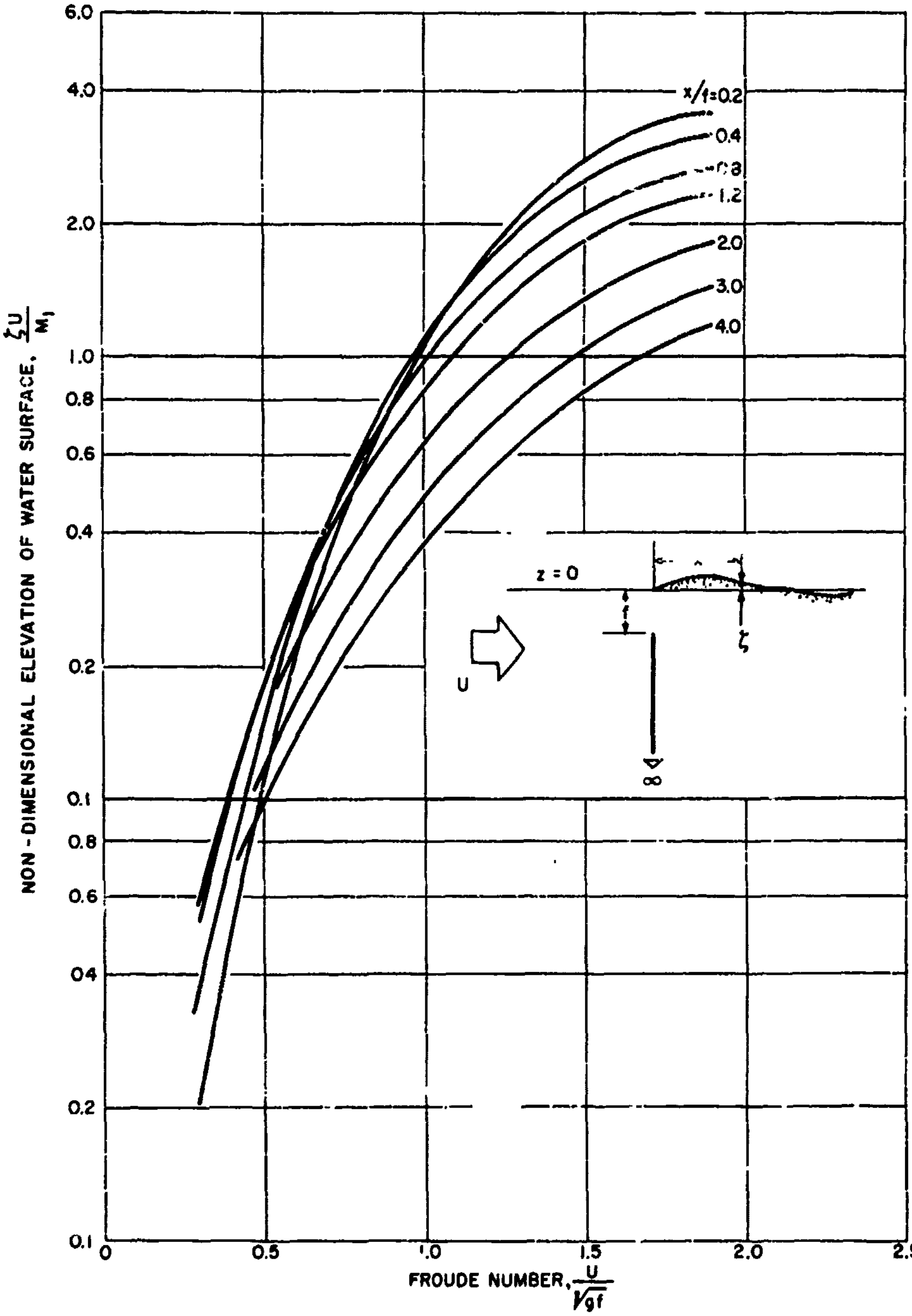

FIGURE 3 – LOCAL DISTURBANCE DUE TO AN INFINITE SOURCE LINE

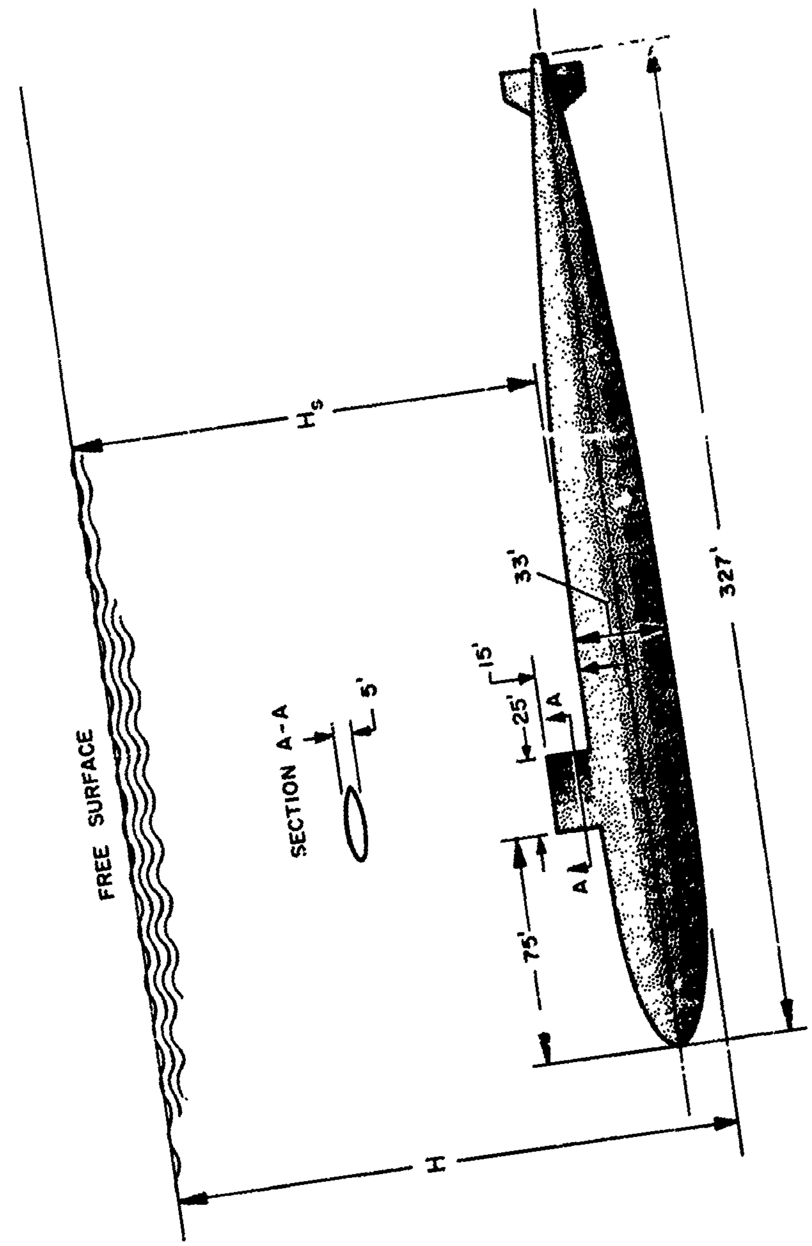

FIGURE 4 - TYPICAL SUBMARINE USED IN SUBMARINE WAVE STUDIES

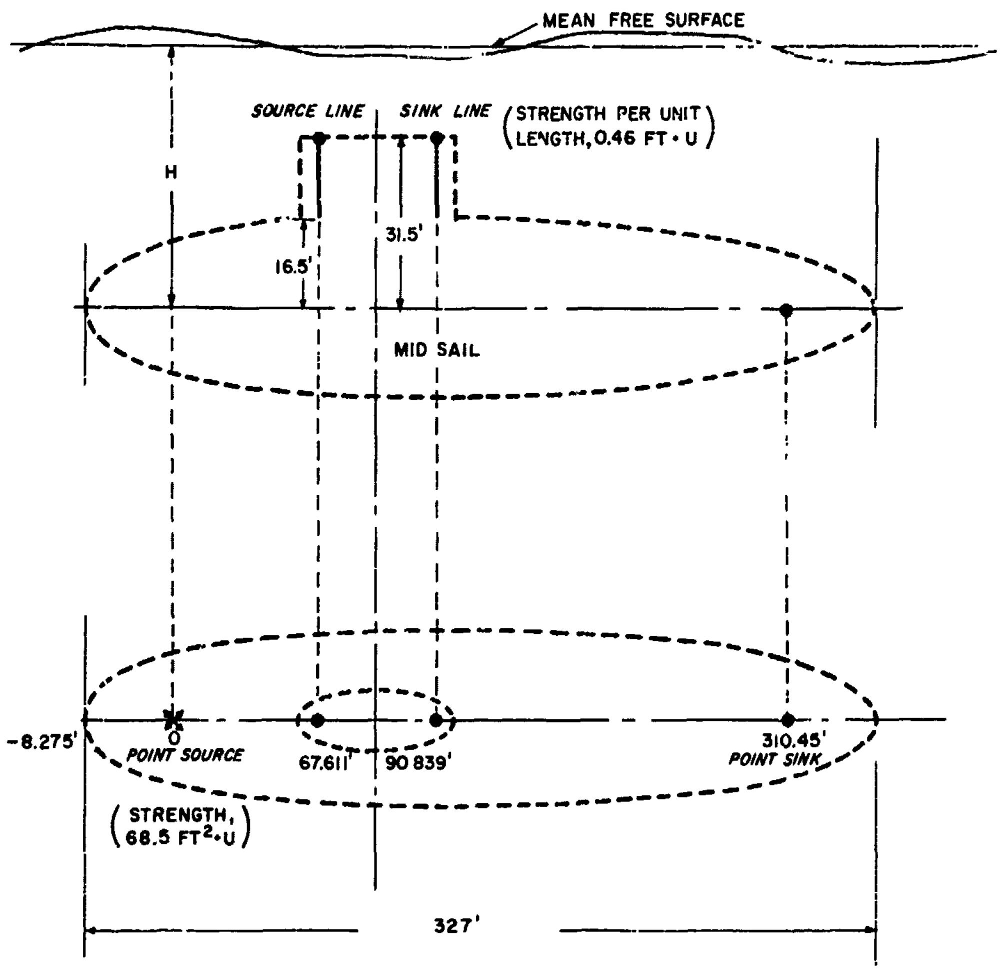

FIGURE 5 - POSITIONS OF SOURCE AND SINK

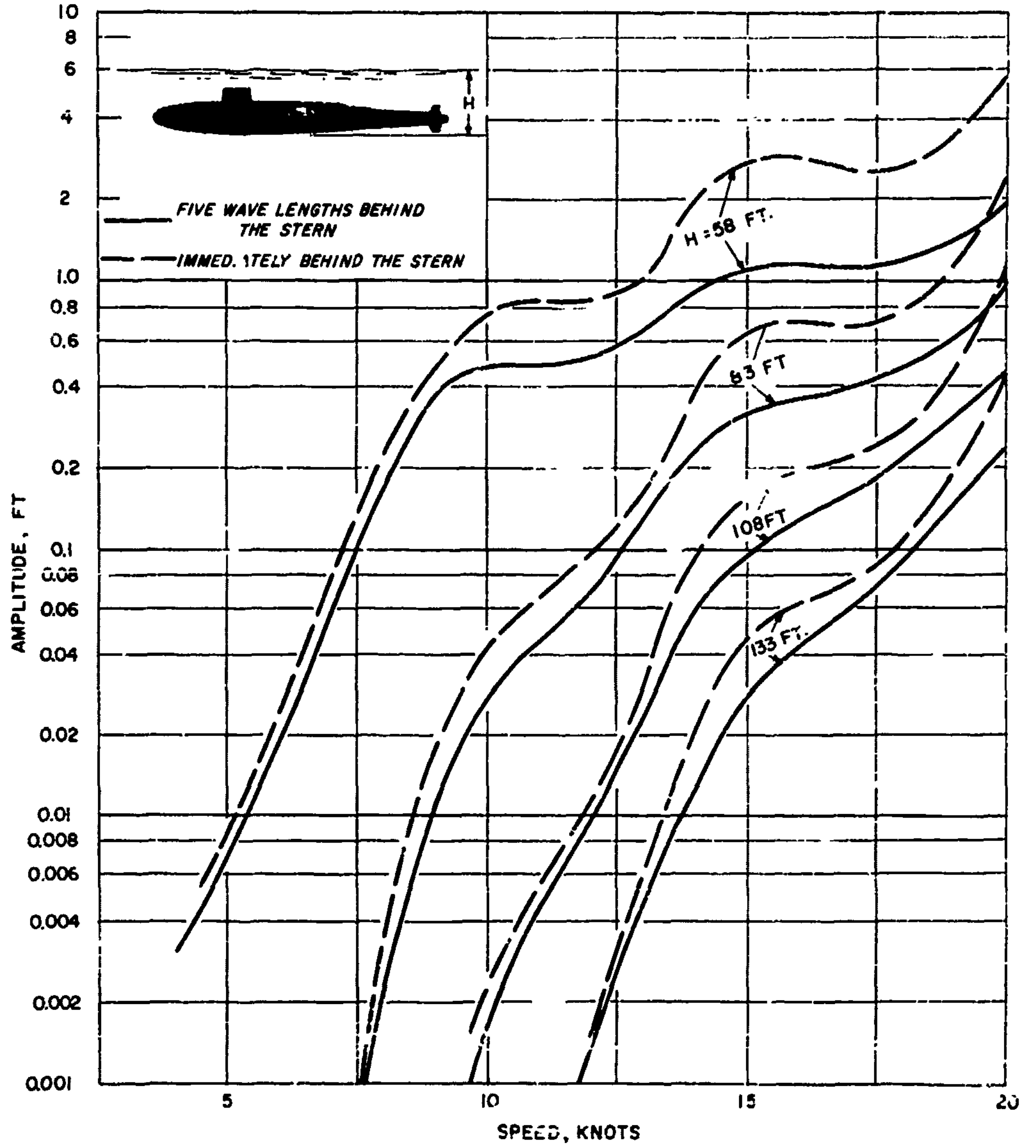

FIGURE 6a – AMPLITUDE OF REGULAR WAVES ON THE CENTERLINE

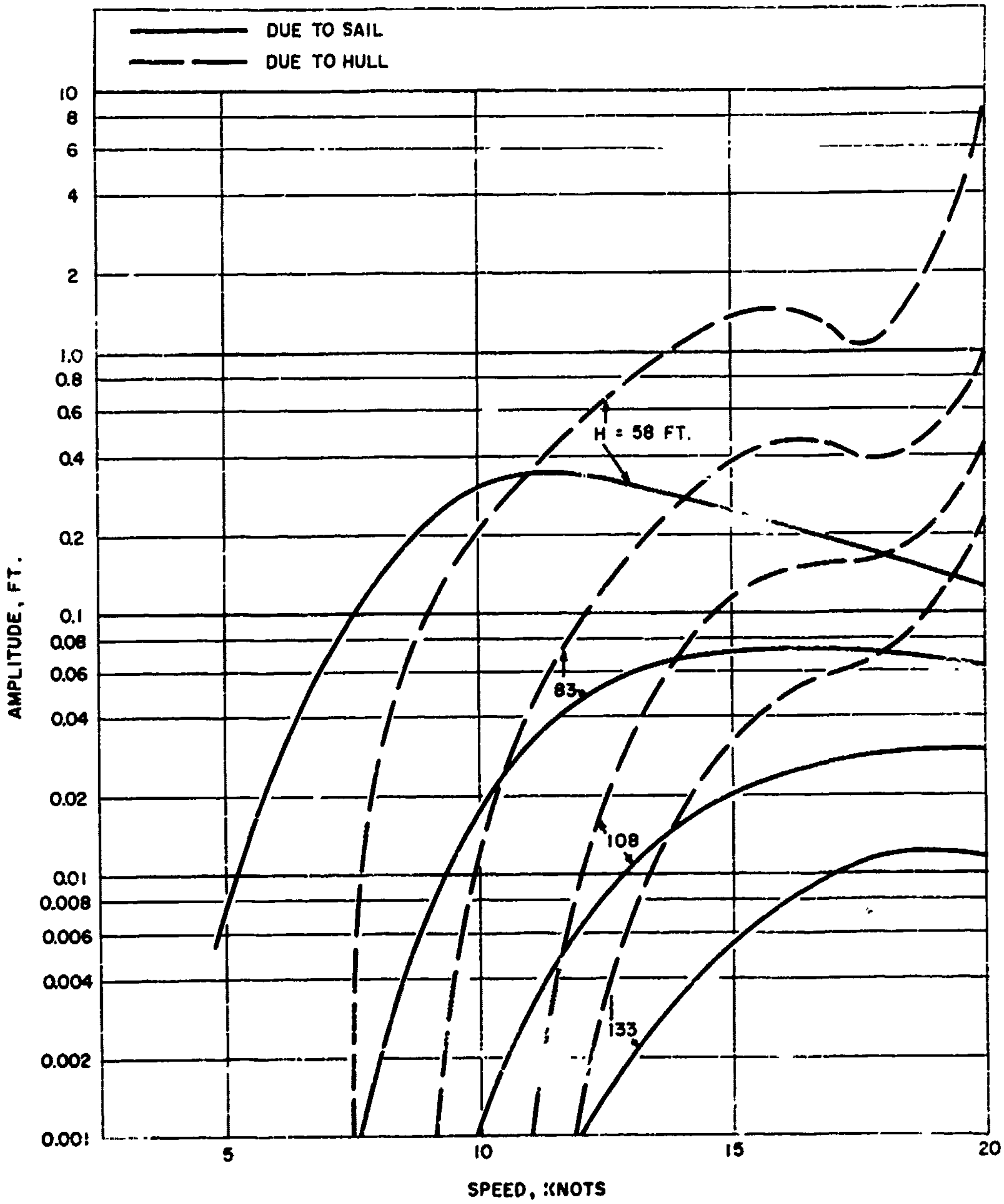

FIGURE 6b – AMPLITUDE OF REGULAR WAVES ON THE CENTERLINE FIVE WAVE LENGTHS AFT OF STERN

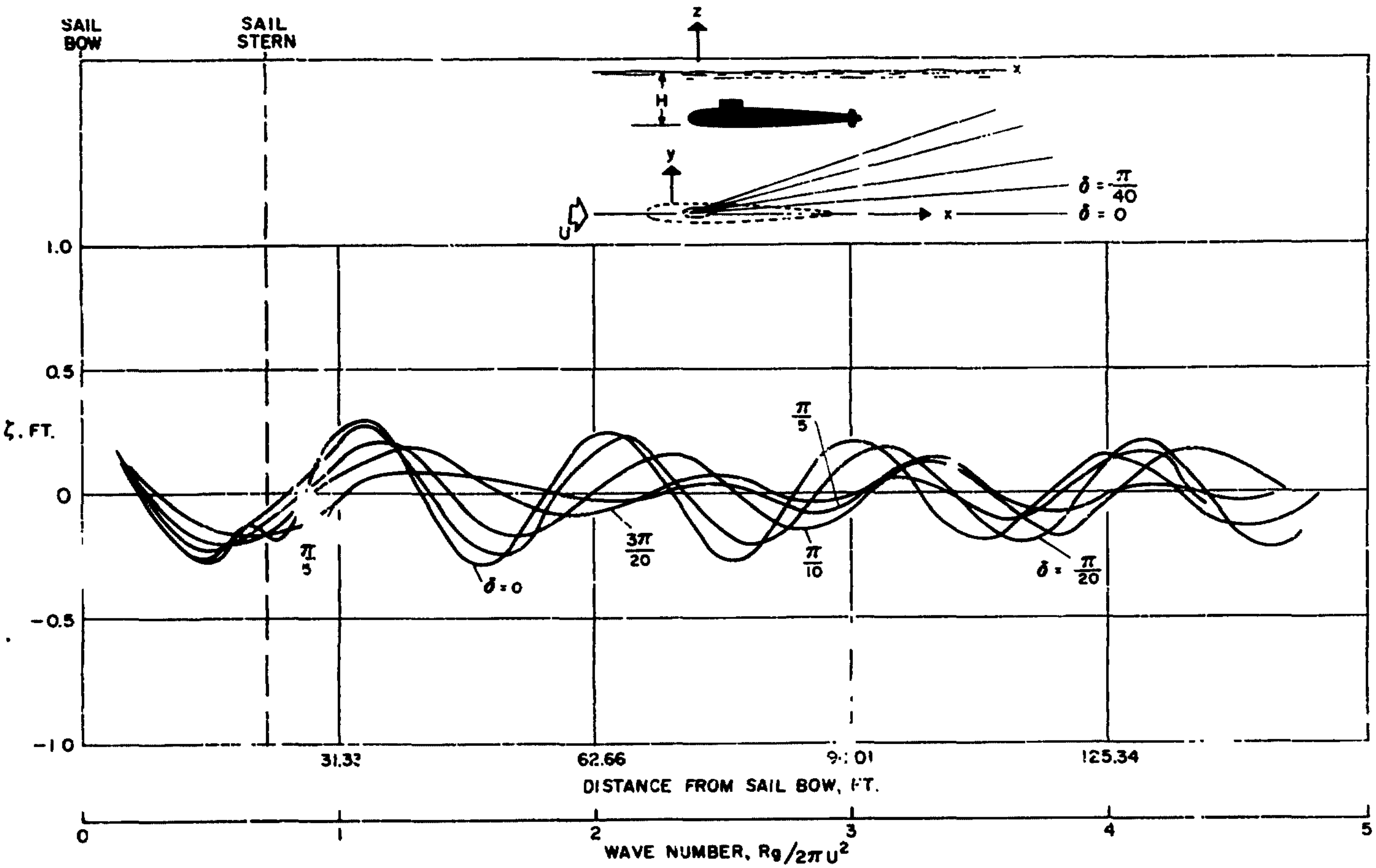

FIGURE 7 – REGULAR WAVES DUE TO SUBMARINE WITH SAIL
U = 7.5 KNOTS
H = 58 FT.

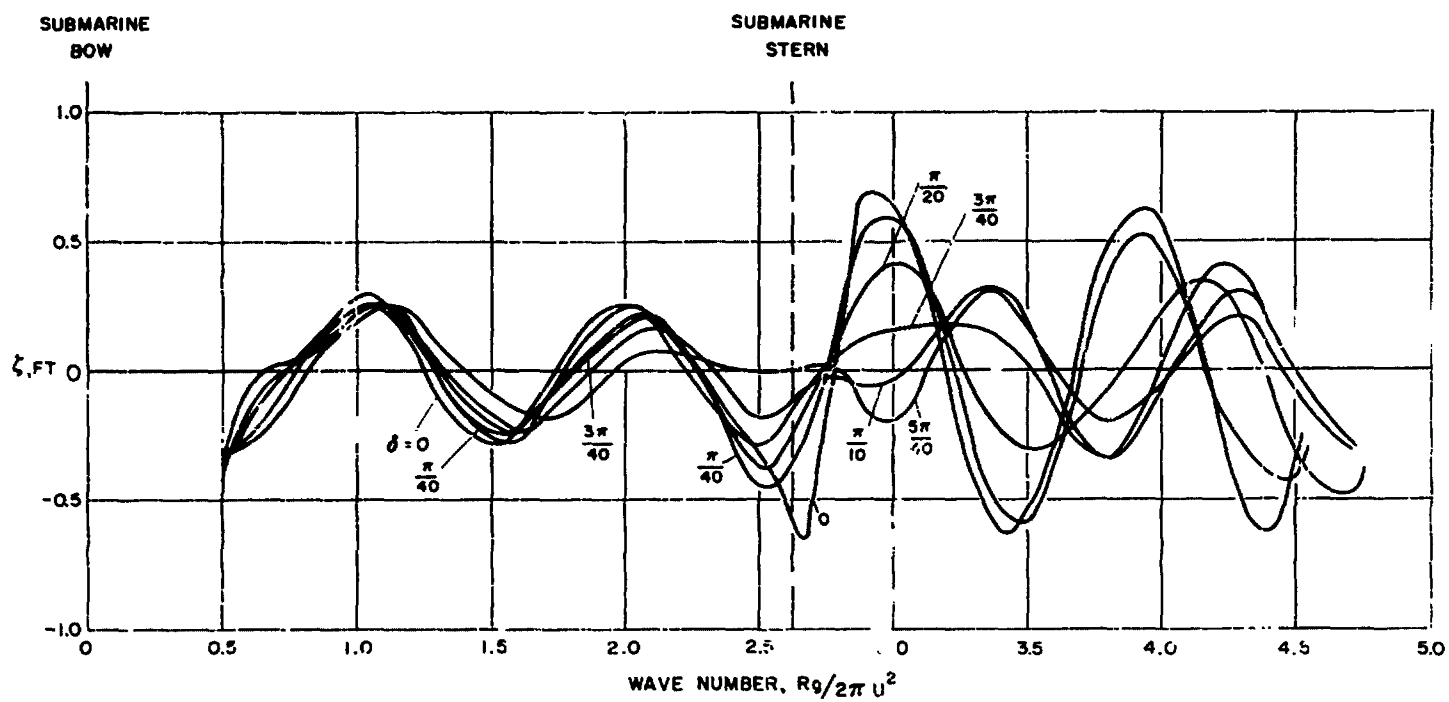

FIGURE 8 – REGULAR WAVES DUE TO SUBMARINE WITH SAIL

[illegible] = 15 KNOTS
[illegible] = 83 FT.

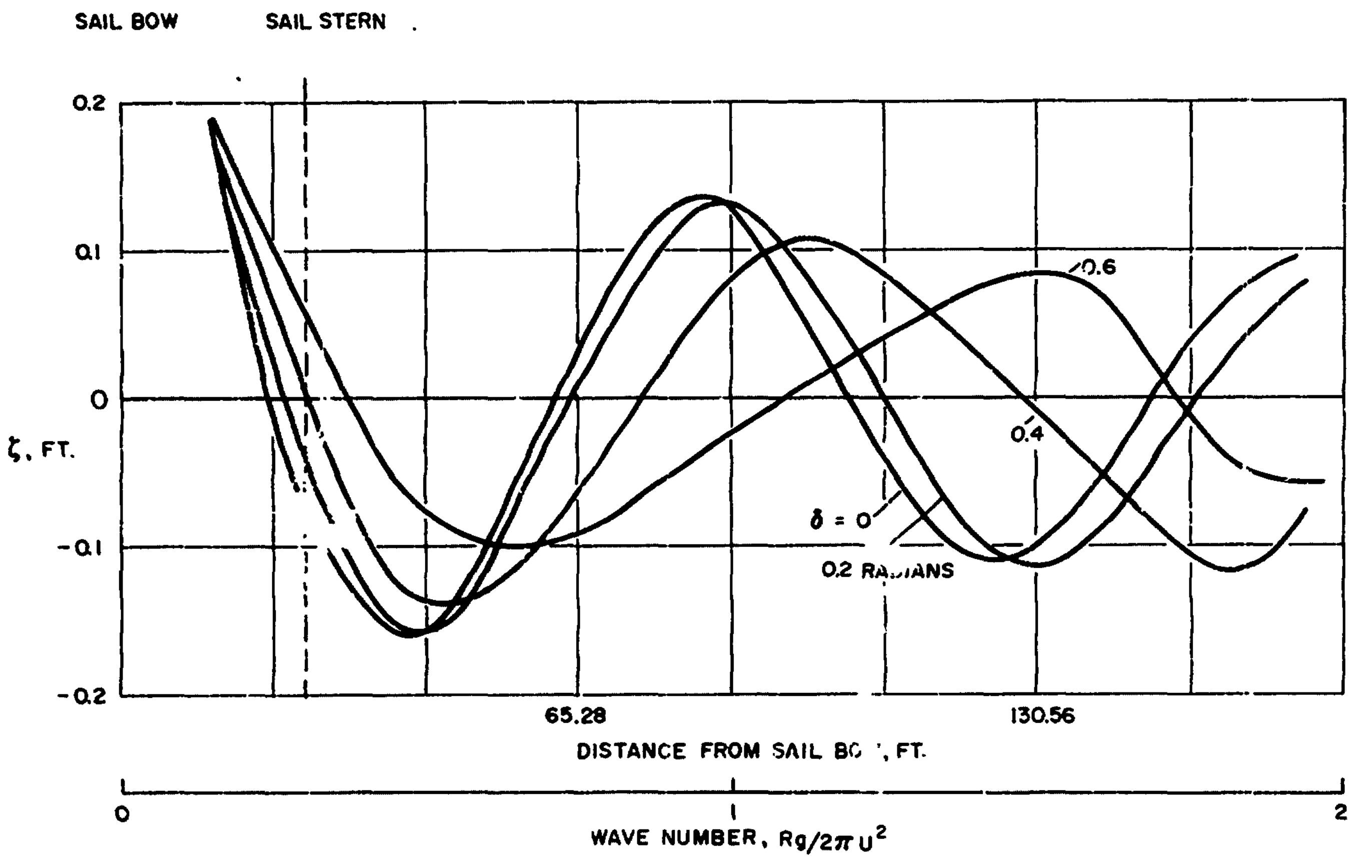

FIGURE 9 - REGULAR WAVF DUE TO SUBMARINE WITH SAIL

U · 12.5 KNOTS

H · 83 FT.

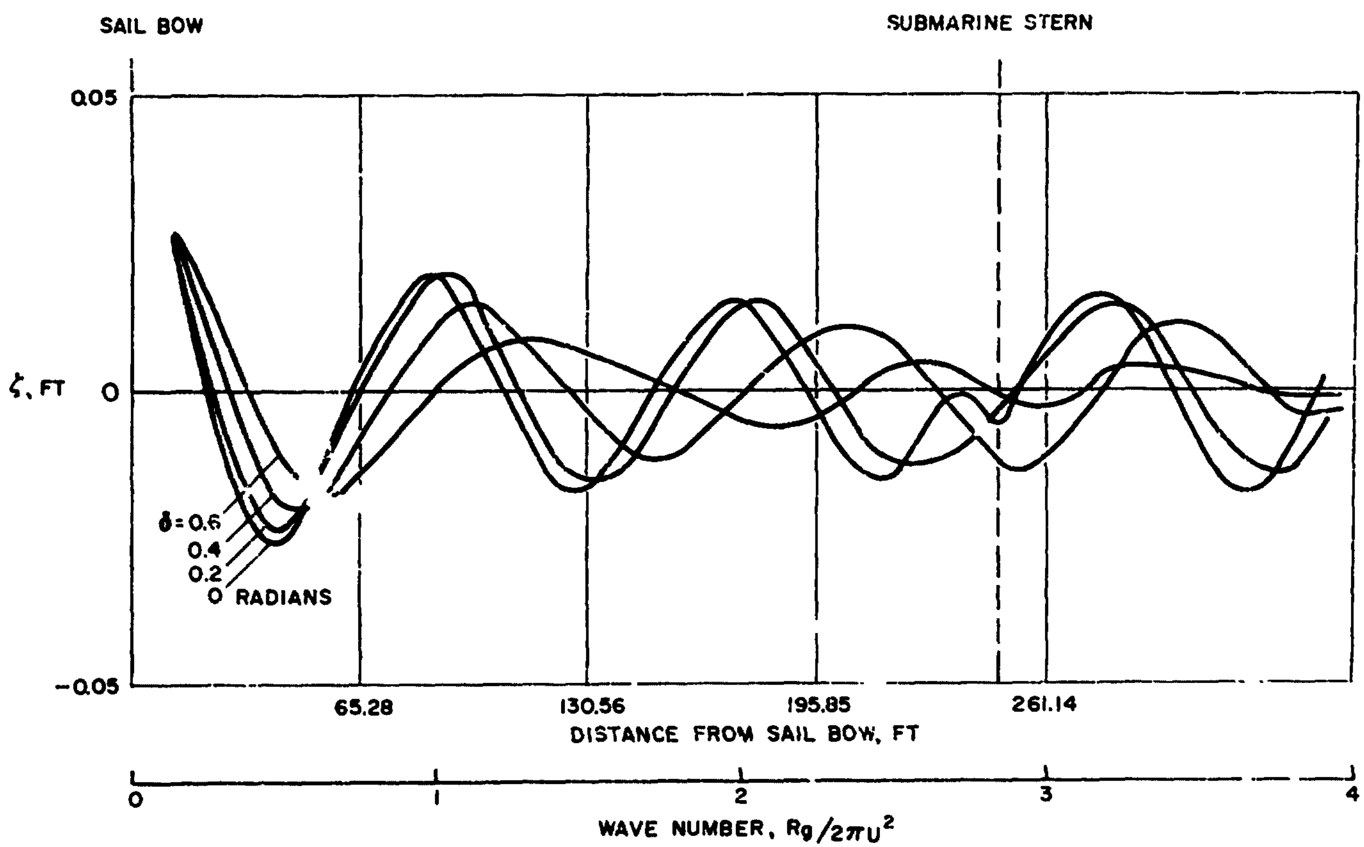

FIGURE 10 – REGULAR WAVE DUE TO SUBMARINE WITH SAIL
U - 12.5 KNOTS
H - 108 FT

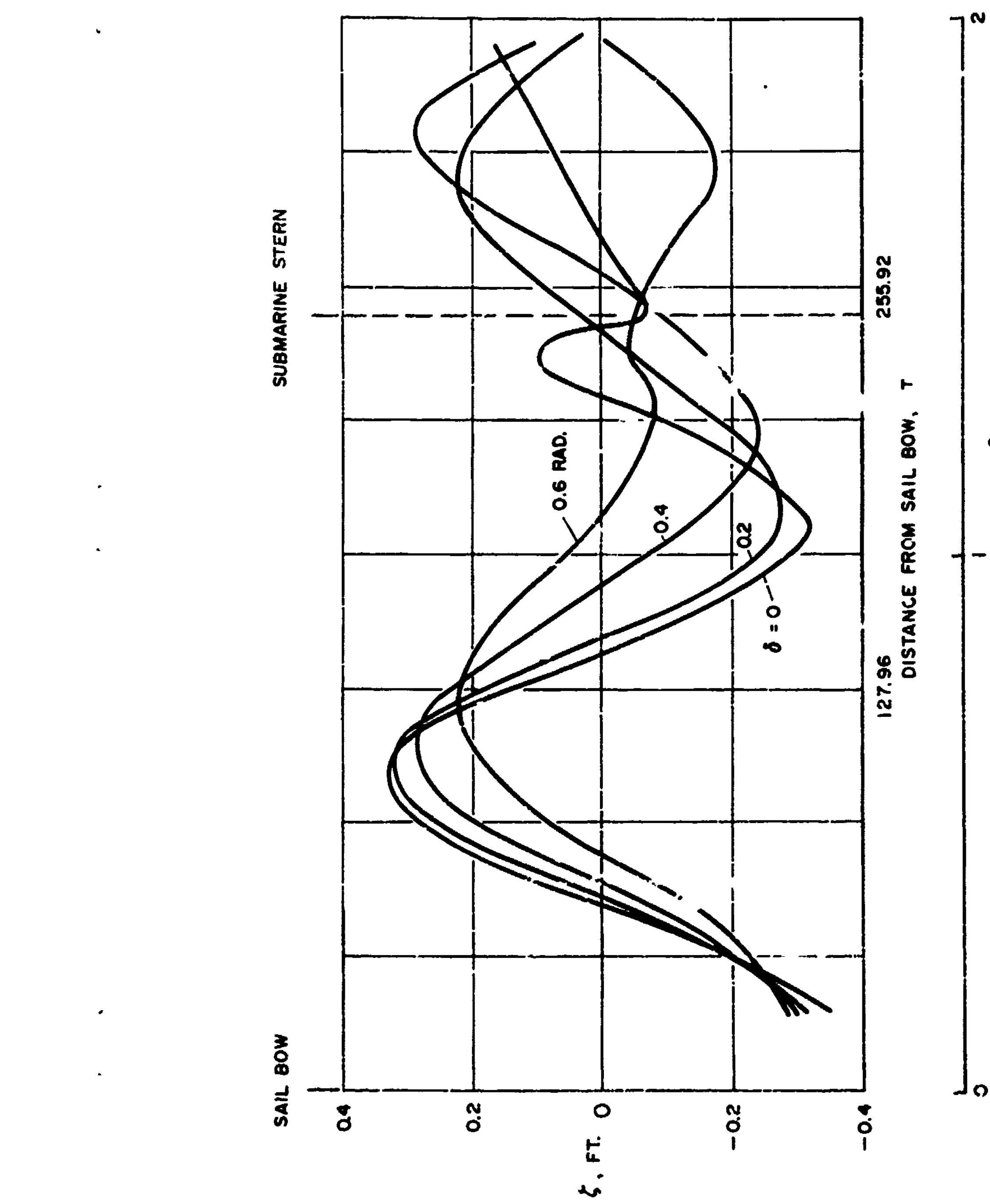

FIGURE 11 – REGULAR WAVE DUE TO SUBMARINE WITH SAIL
U = 17.5 KNOTS
H = 10.8 FT.

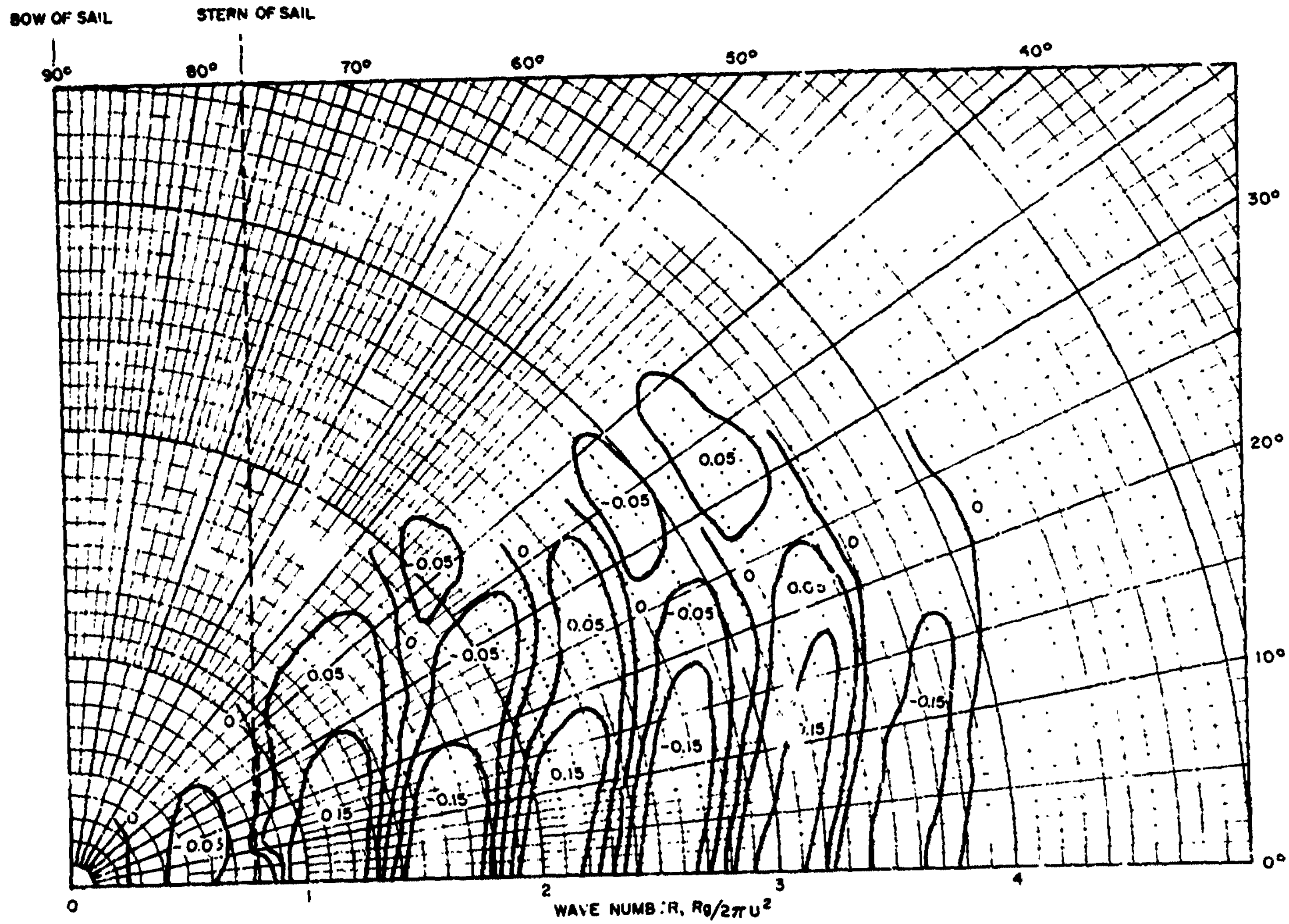

FIGURE 12 – CONTOURS OF WAVE HEIGHT (FT) DUE TO SUBMARINE NEAR THE SAIL
U = 7.5 KNOTS, H = 58 FT

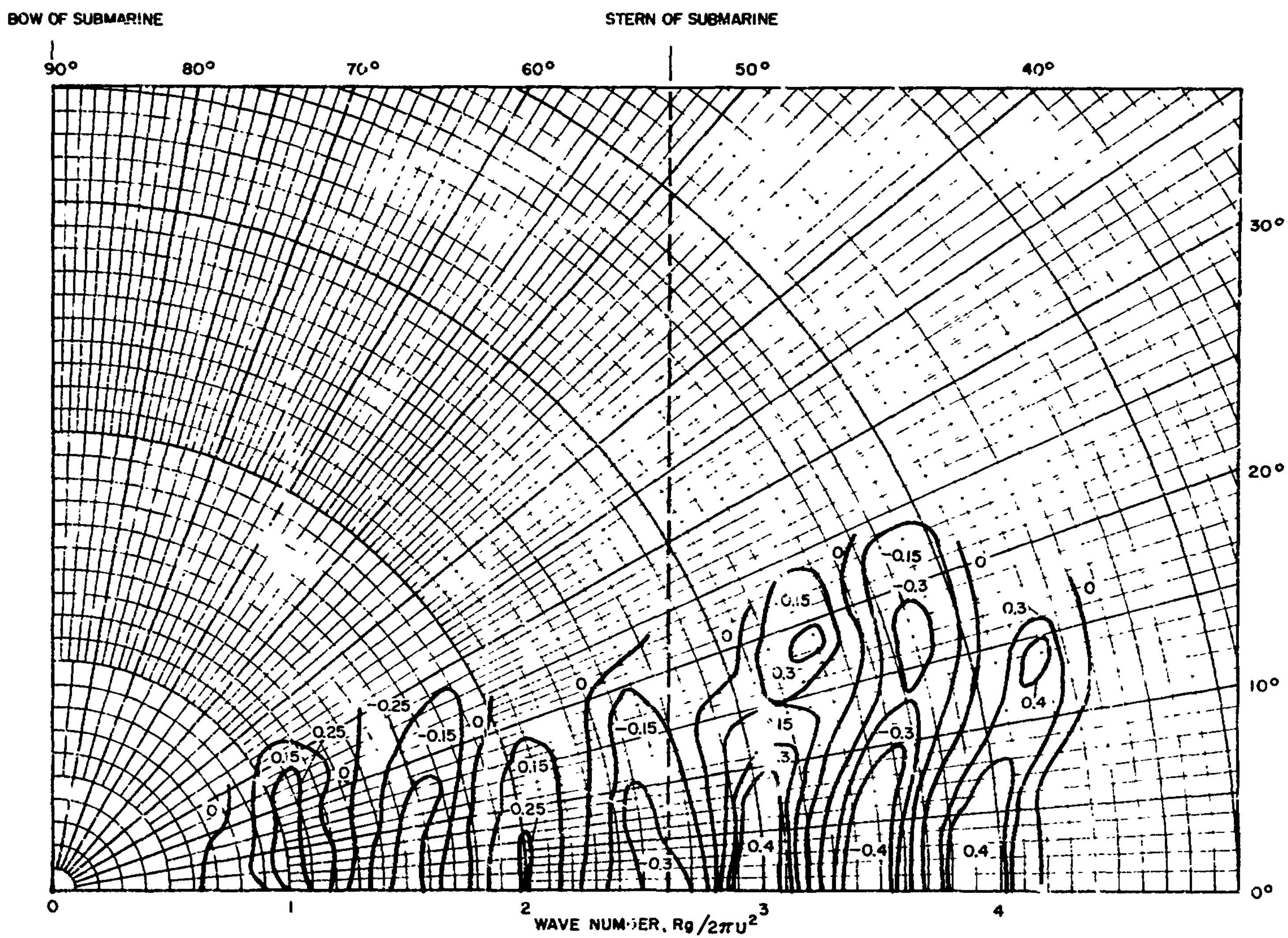

FIGURE 13 – CONTOURS OF WAVE HEIGHT (FT) DUE TO SUBMARINE
U = 15 KNOTS, H = 83 FT

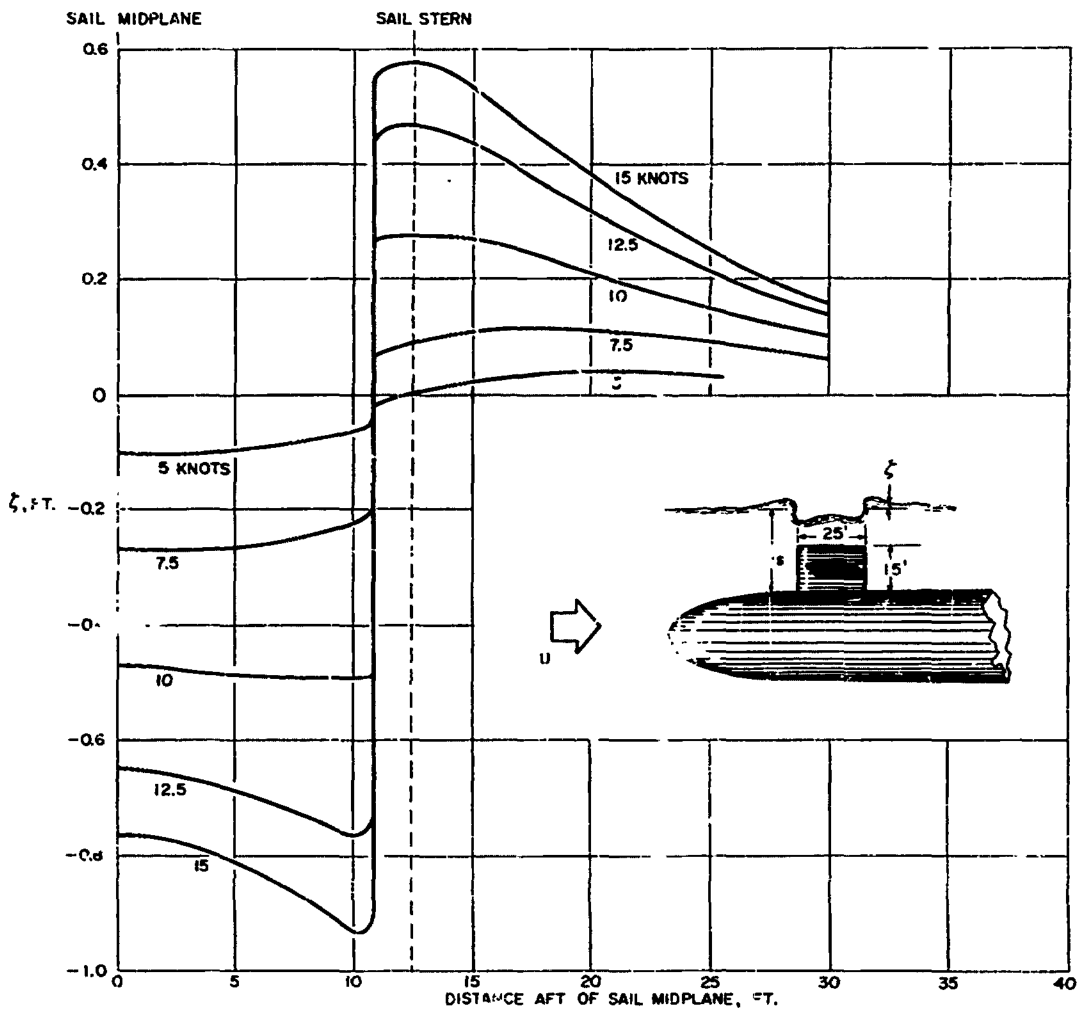

FIGURE 14 – LOCAL DISTURBANCE ALONG CENTERLINE DUE TO SAIL ALONE

$H_s = 25$ FT.

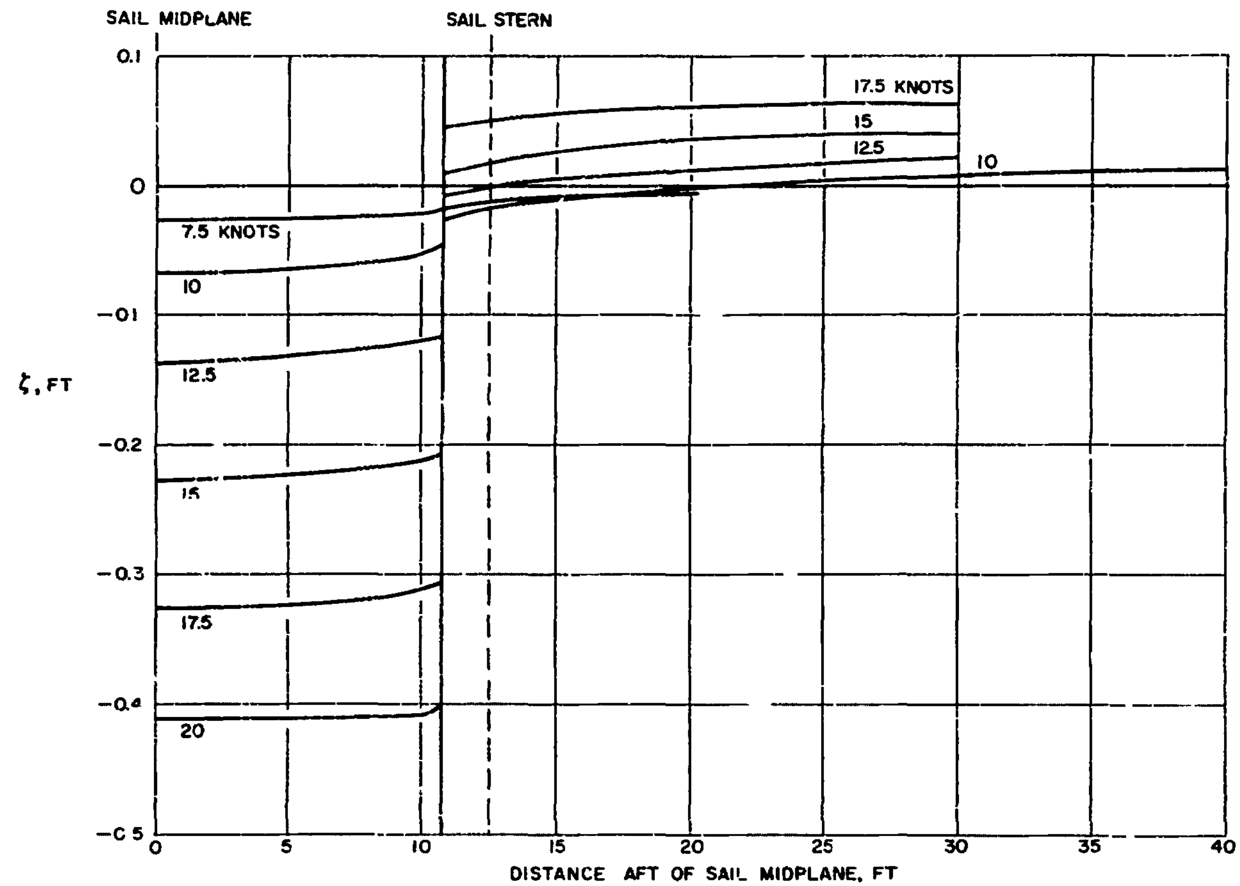

FIGURE 15 – LOCAL DISTURBANCE ALONG CENTERLINE DUE TO SAIL ALONE

$H_s = 50$ FT

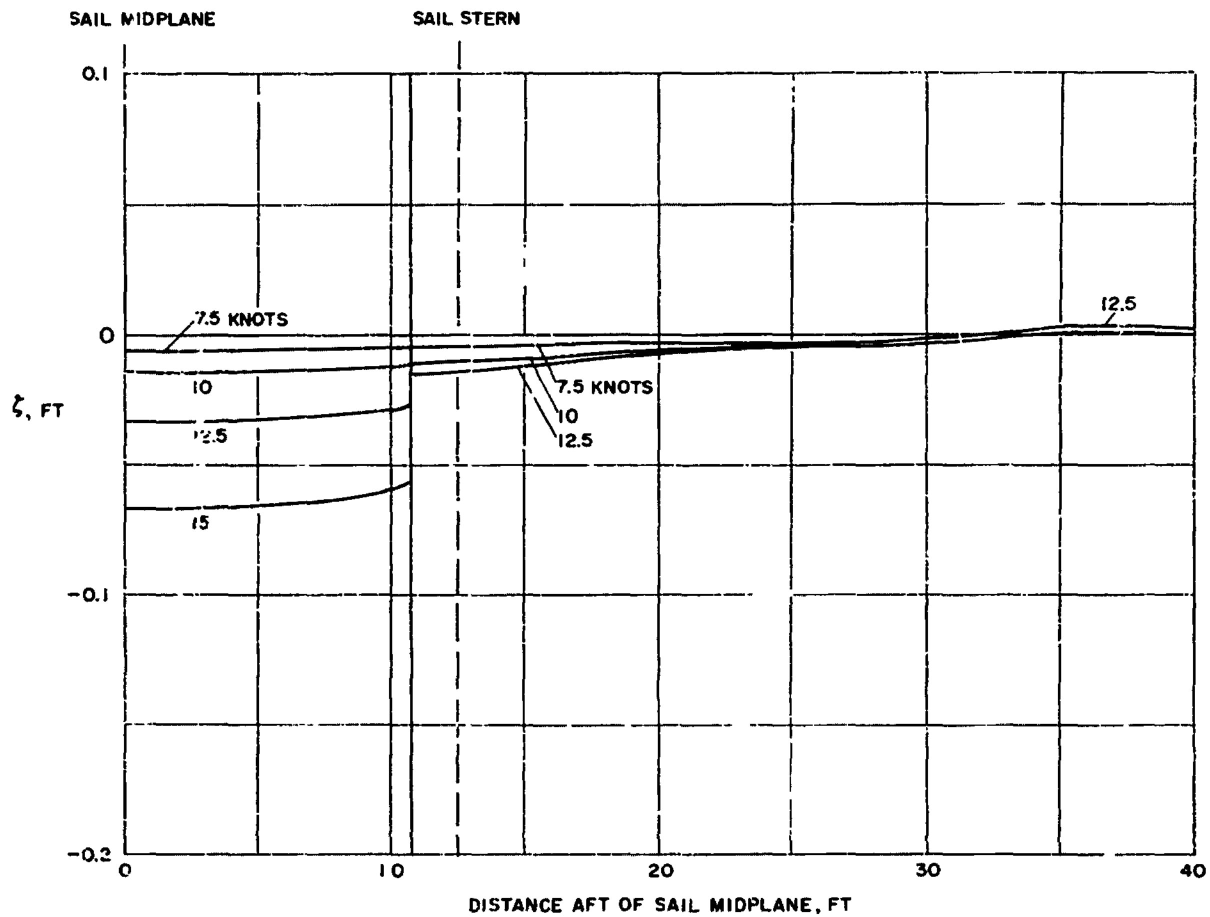

FIGURE 16 – LOCAL DISTURBANCE ALONG CENTERLINE DUE TO SAIL ALONE
H_s = 75 FT.

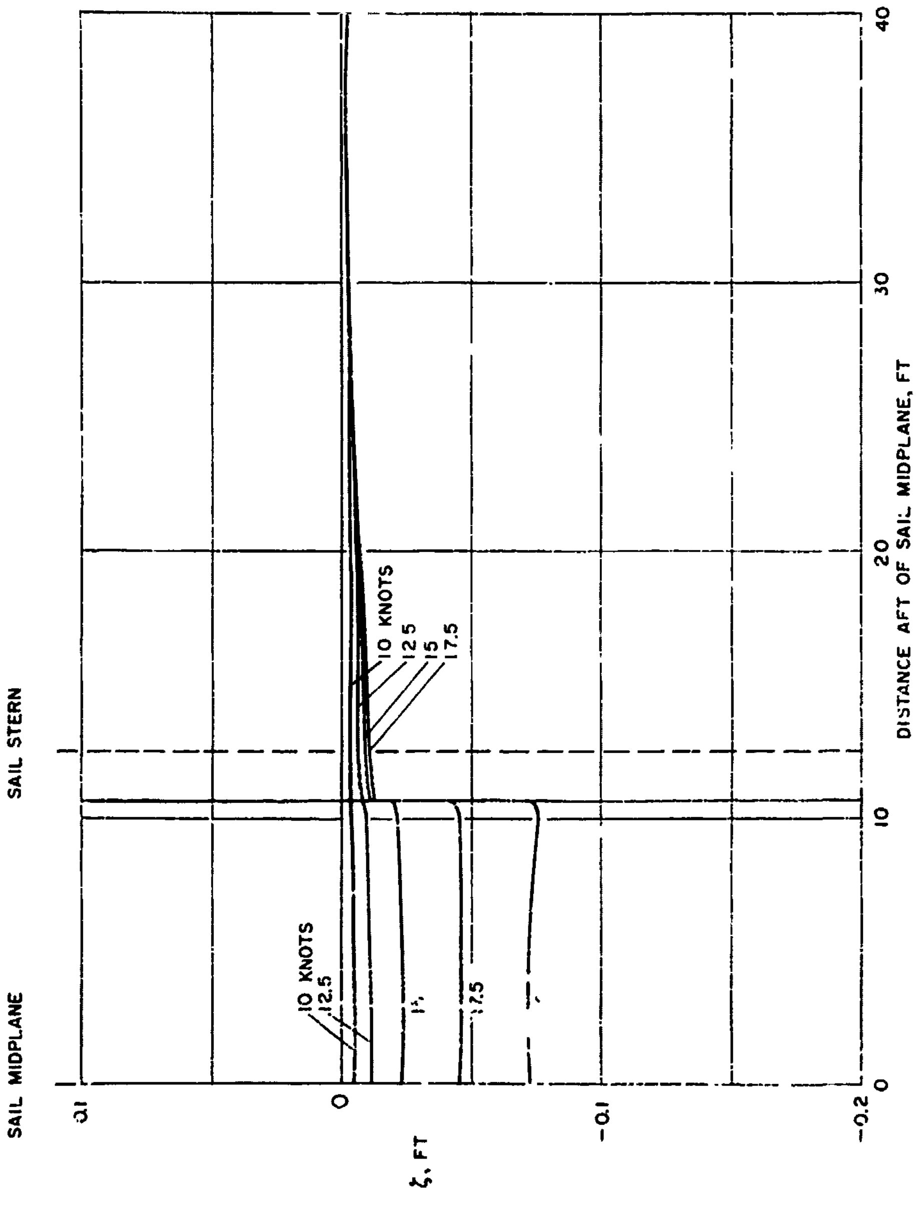

FIGURE 17 - LOCAL DISTURBANCE ALONG CENTERLINE DUE TO SAIL ALONE

$H_s = 100$ FT

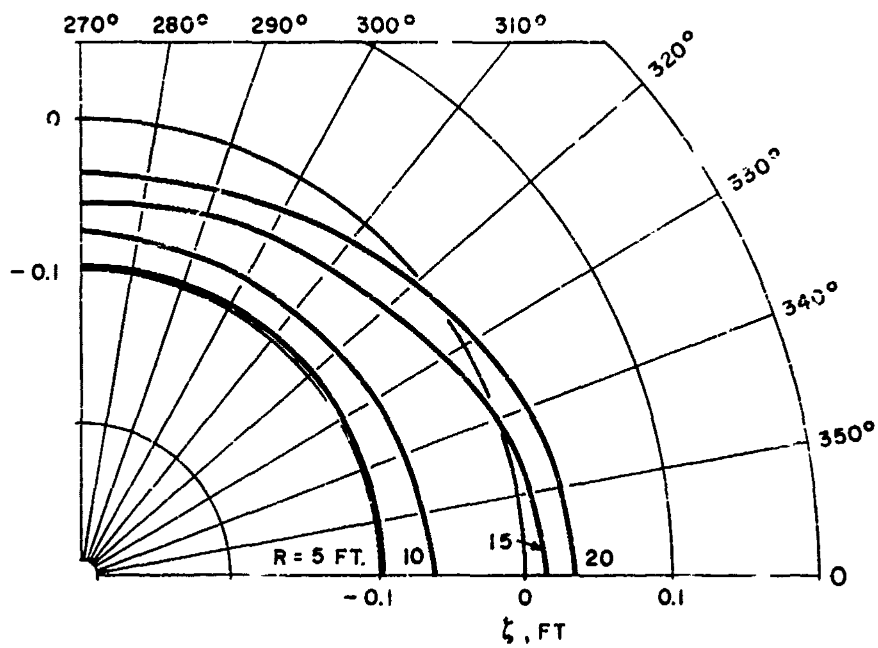

FIGURE 18 -- LOCAL DISTURBANCE DUE TO SAIL ALONE

U = 5 KNOTS, H_s = 25 FT

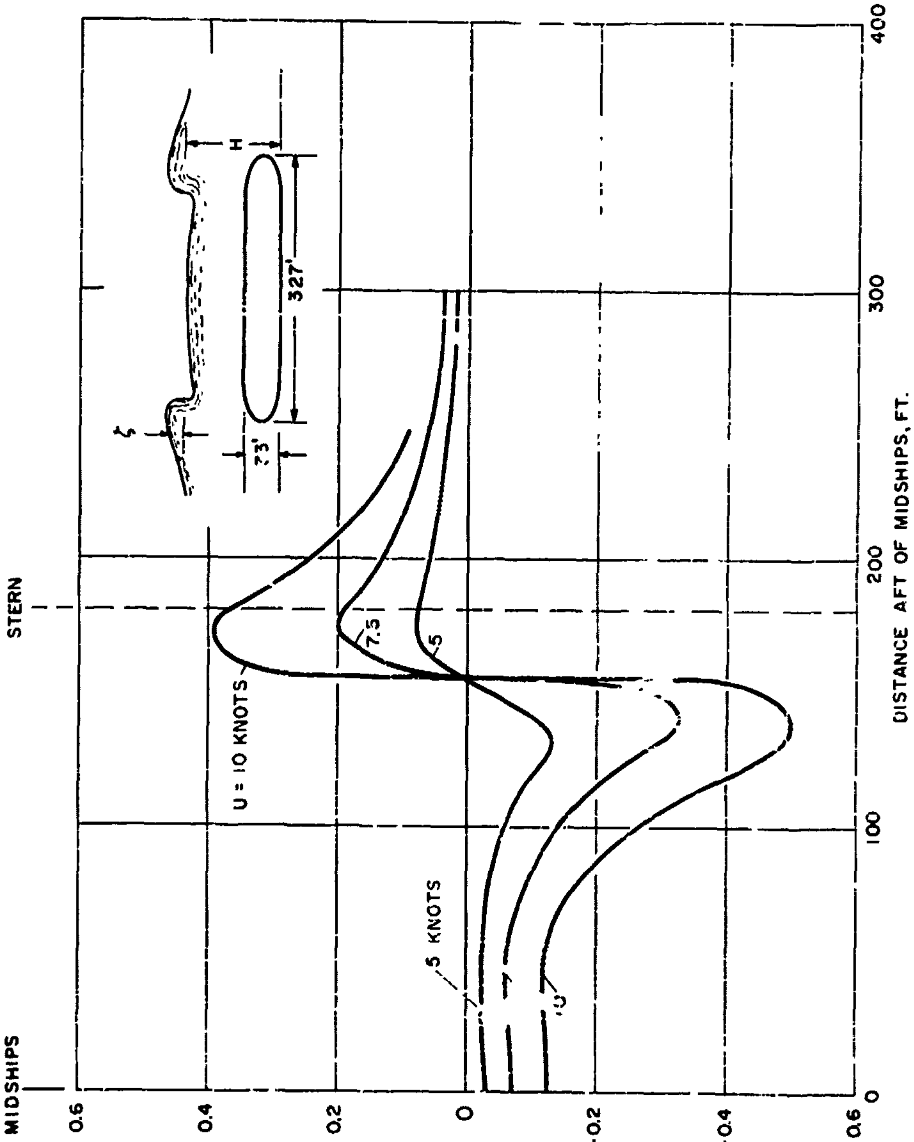

FIGURE 19 – LOCAL DISTURBANCE ALONG CENTERLINE DUE TO HULL ALONE

H = 53 FT.

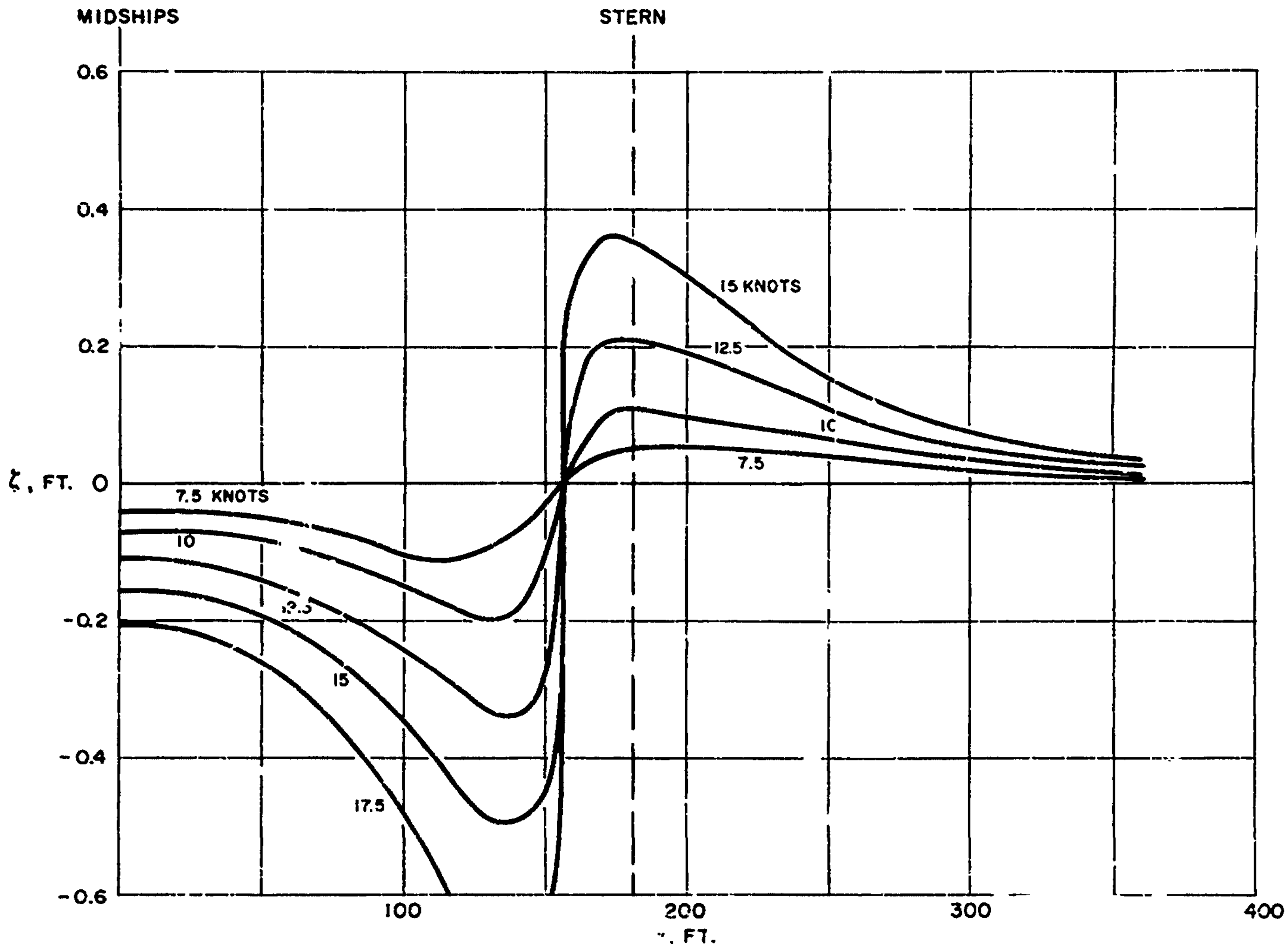

FIGURE 20 - LOCAL DISTURBANCE ALONG CENTERLINE DUE TO HULL ALONE

H = 83 FT.

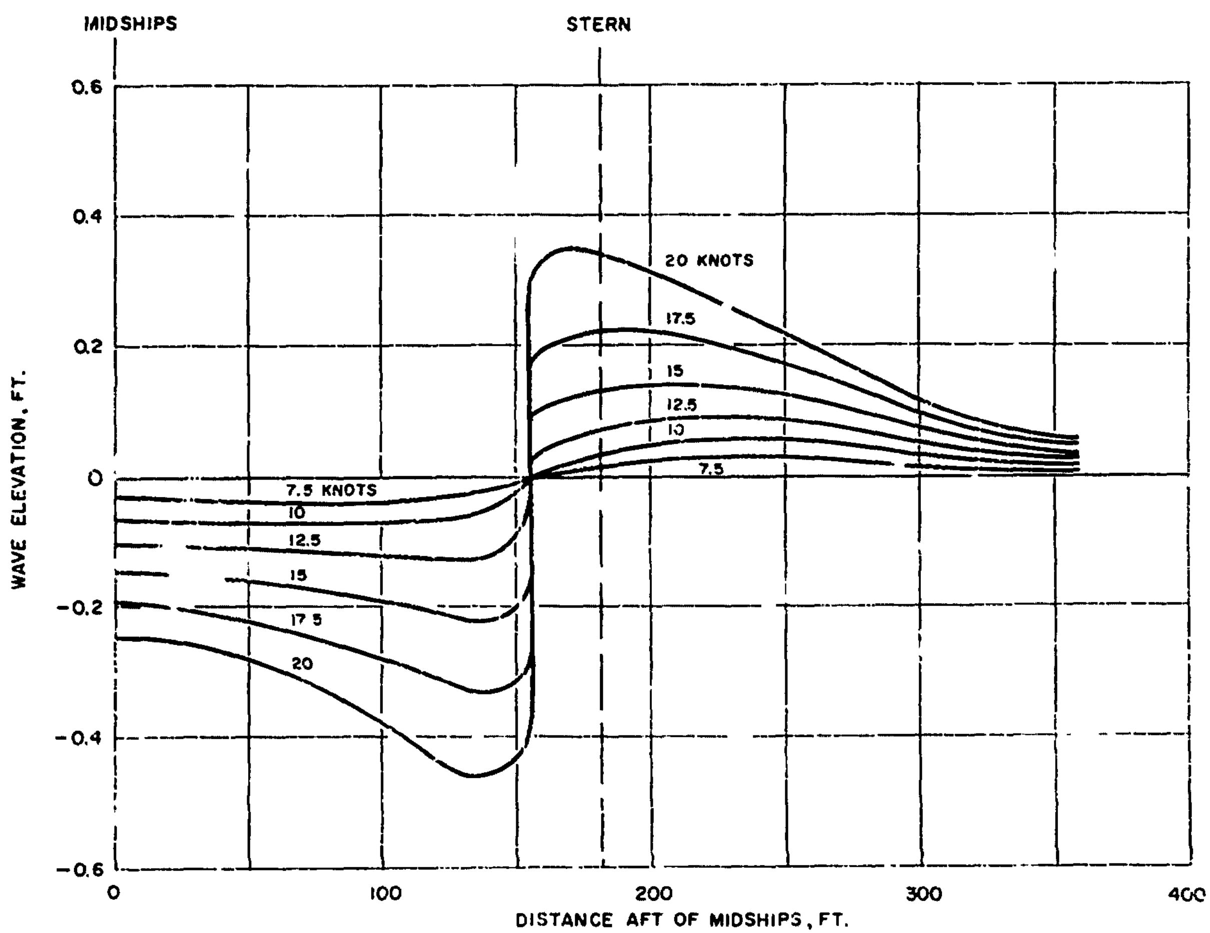

FIGURE 2 – LOCAL DISTURBANCE ALONG CENTERLINE DUE TO HULL ALONE

H = 108 FT.

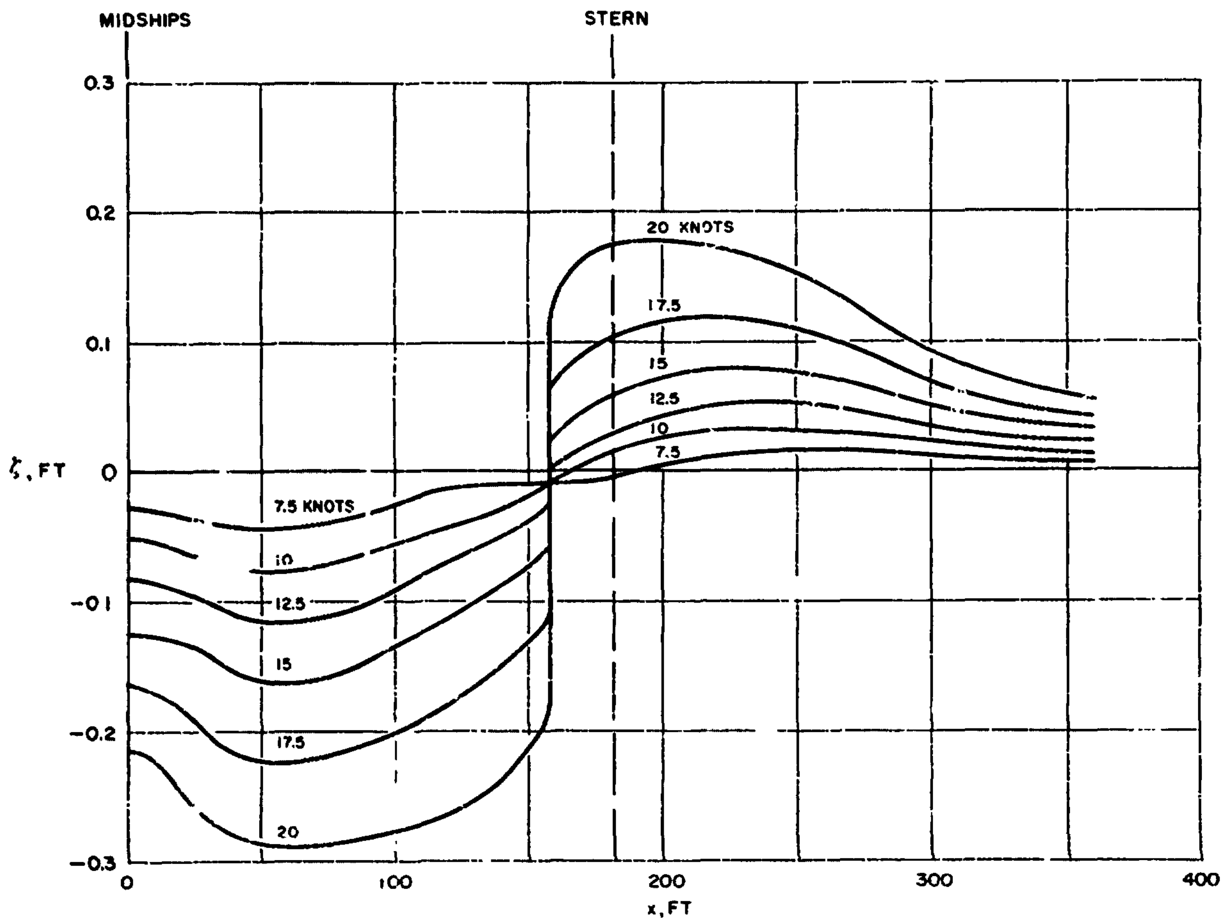

FIGURE 22 - LOCAL DISTURBANCE ALONG CENTERLINE DUE TO HULL ALONE
H = 133 FT

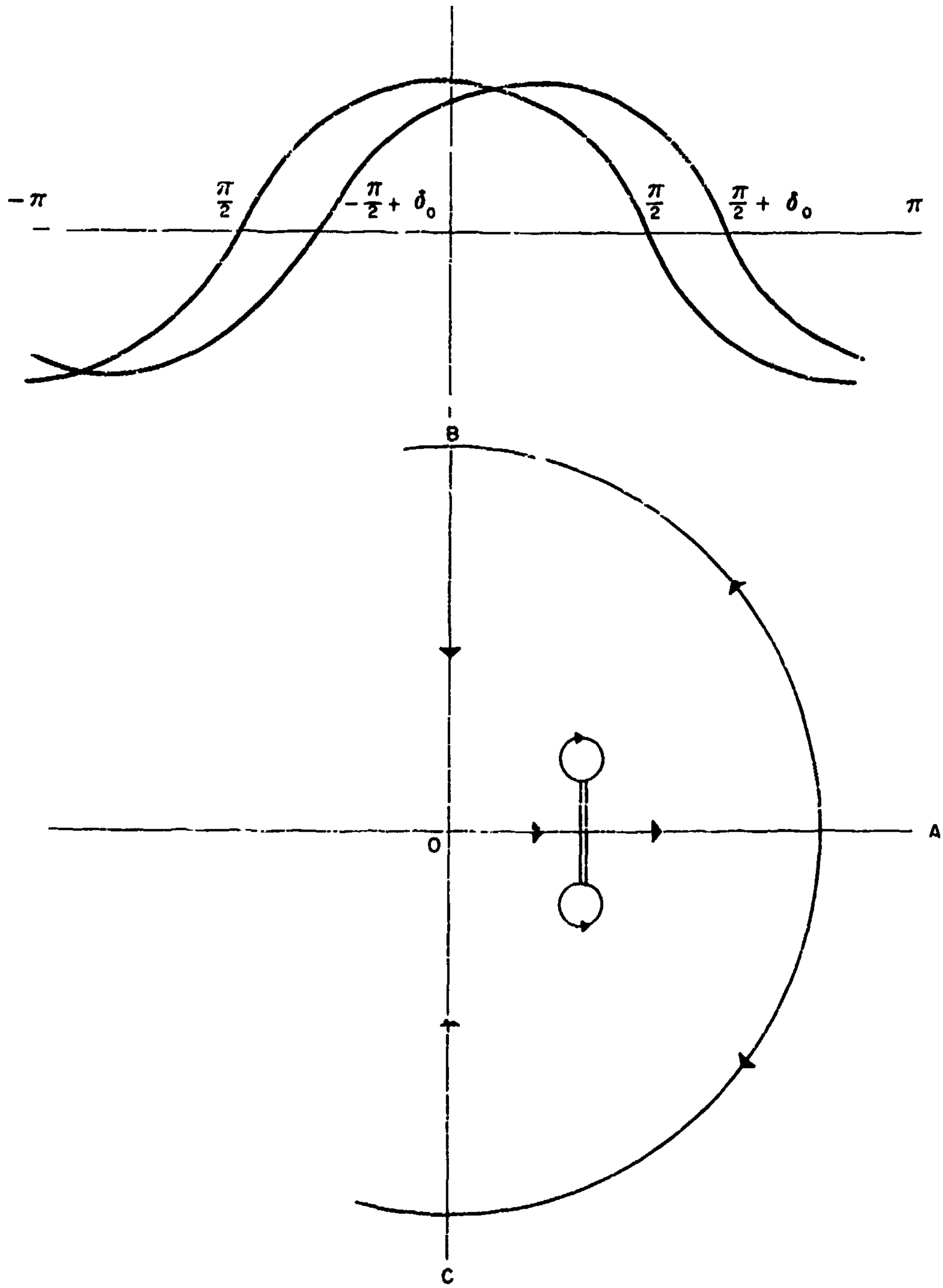

FIGURE A1 - CONTOUR OF INTEGRATION

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